Stochastic Modelling of Counterparty Credit Risk, CVA and DVA – with a Glimpse at the New Regulatory Framework Basel III

Frank Oertel

University of Southampton
School of Mathematics

Advances in Mathematics of Finance
6th AMaMeF and Banach Center Conference, Warsaw
10.06.2013 - 15.06.2013

Disclaimer

This presentation and associated materials are provided for informational and educational purposes. Views expressed in this work are the author's views.

Disclaimer

This presentation and associated materials are provided for informational and educational purposes. Views expressed in this work are the author's views.

In particular, this presentation is by no means linked to any present and future wording regarding global regulation of CCR including EMIR and CRR (CRD IV).

Contents

1. Simultaneous defaults and total bilateral counterparty credit risk

2. Total bilateral valuation adjustment

3. Unilateral CVA and Basel III

Counterparty credit risk - Outline I

Suppose there are two parties who are trading a portfolio of OTC derivative contracts such as, e. g., a portfolio of CDSs. (Bilateral) counterparty credit risk (CCR) is the risk that *at least one* of those two parties in that derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments to its counterpart.

Counterparty credit risk - Outline I

Suppose there are two parties who are trading a portfolio of OTC derivative contracts such as, e. g., a portfolio of CDSs. (Bilateral) counterparty credit risk (CCR) is the risk that *at least one* of those two parties in that derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments to its counterpart.

Future cashflow exchanges are not known with certainty today. The main feature that distinguishes CCR from the risk of a standard loan is the uncertainty of the exposure at any future date. Hence, regarding the modelling of the exposure a simulation of future cashflow exchanges is necessary.

Counterparty credit risk - Outline II

What has to be analysed in CCR?

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the "exposure", i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the "exposure", i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
- Calculation of the default probability of each of the both trading parties (can be derived e. g. from CDS prices in the market);

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the “exposure”, i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
- Calculation of the default probability of each of the both trading parties (can be derived e. g. from CDS prices in the market);
- Analysis of a dependence between exposure and default time (“Wrong-Way Risk” and “Right-Way-Risk”);

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the “exposure”, i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
- Calculation of the default probability of each of the both trading parties (can be derived e. g. from CDS prices in the market);
- Analysis of a dependence between exposure and default time (“Wrong-Way Risk” and “Right-Way-Risk”);
- Calculation of a value adjustment on top of the “CCR free” market value of a transaction, implying a “pricing of CCR”;

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the “exposure”, i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
- Calculation of the default probability of each of the both trading parties (can be derived e. g. from CDS prices in the market);
- Analysis of a dependence between exposure and default time (“Wrong-Way Risk” and “Right-Way-Risk”);
- Calculation of a value adjustment on top of the “CCR free” market value of a transaction, implying a “pricing of CCR”;
- Basel III capital charge for the “value adjustment volatility risk”;

Counterparty credit risk - Outline II

What has to be analysed in CCR?

- Modelling of the "exposure", i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
- Calculation of the default probability of each of the both trading parties (can be derived e. g. from CDS prices in the market);
- Analysis of a dependence between exposure and default time ("Wrong-Way Risk" and "Right-Way-Risk");
- Calculation of a value adjustment on top of the "CCR free" market value of a transaction, implying a "pricing of CCR";
- Basel III capital charge for the "value adjustment volatility risk";
- Impact of central clearing through a CCP.

CCR - The framework I

Consider two arbitrary parties who trade with each other according to an underlying financial contract - “party 0” and “party 2”, say. Let $k \in \{0, 2\}$ and $T > 0$ the final maturity of this financial contract.

CCR - The framework I

Consider two arbitrary parties who trade with each other according to an underlying financial contract - “party 0” and “party 2”, say. Let $k \in \{0, 2\}$ and $T > 0$ the final maturity of this financial contract.

- Let $X = (X(t))_{0 \leq t \leq T}$ denote a stochastic process describing a cash flow between party 0 and party 2 (or a random sequence of prices). If, at time t, X_t is seen from the point of view of party k, we denote its value equivalently as $X_t(k)$ or $X_t(k, 2-k)$ or $X_k(t)$ - depending on its eligibility.

CCR - The framework I

Consider two arbitrary parties who trade with each other according to an underlying financial contract - “party 0” and “party 2”, say. Let $k \in \{0, 2\}$ and $T > 0$ the final maturity of this financial contract.

- Let $X = (X(t))_{0 \leq t \leq T}$ denote a stochastic process describing a cash flow between party 0 and party 2 (or a random sequence of prices). If, at time t, X_t is seen from the point of view of party k, we denote its value equivalently as $X_t(k)$ or $X_t(k, 2 - k)$ or $X_k(t)$ - depending on its eligibility.
- We will make use of the important notation $Y_t(k \mid l)$ to describe a random cash flow amount from the point of view of party k at time t contingent on the default of party l, where $l \in \{0, 2\}$.

CCR - The framework II

- "Contingent cash flows" between two trading parties can be embedded into the model of a generalised weighted and directed Erdős-Rényi random graph ("network"), consisting of counterparties as nodes and current exposures as edges, leading to matrix-valued stochastic processes of type $(\omega, t) \mapsto \mathbf{A}(\omega, t)$, where $\mathbf{A}(\omega, t)_{k,l} := Y_t(k \mid l)(\omega)$ describes a contingent cash flow between the nodes k and l, given that l will default until T (with probability 1).

CCR - The framework II

- "Contingent cash flows" between two trading parties can be embedded into the model of a generalised weighted and directed Erdős-Rényi random graph ("network"), consisting of counterparties as nodes and current exposures as edges, leading to matrix-valued stochastic processes of type $(\omega, t) \mapsto \mathbf{A}(\omega, t)$, where $\mathbf{A}(\omega, t)_{k,l} := Y_t(k \mid l)(\omega)$ describes a contingent cash flow between the nodes k and l, given that l will default until T (with probability 1).
- Clearing through a central counterparty (CCP) changes this graph to a (possibly disconnected) tree.

CCR - The framework II

- "Contingent cash flows" between two trading parties can be embedded into the model of a generalised weighted and directed Erdős-Rényi random graph ("network"), consisting of counterparties as nodes and current exposures as edges, leading to matrix-valued stochastic processes of type $(\omega, t) \mapsto \mathbf{A}(\omega, t)$, where $\mathbf{A}(\omega, t)_{k,l} := Y_t(k \mid l)(\omega)$ describes a contingent cash flow between the nodes k and l, given that l will default until T (with probability 1).
- Clearing through a central counterparty (CCP) changes this graph to a (possibly disconnected) tree.
- Notice that the permutation $s : \{0, 2\} \longrightarrow \{0, 2\}, k \mapsto 2 - k$ is bijective. It satisfies $s \circ s = s$. (If s should permute the numbers 1 and 2 instead, then put $s(k) := 3 - k$.)

Modelling of default times

Let τ_0 denote the default time of the investor and τ_2 the default time of the counterparty. Suppose that the underlying financial market model is arbitrage-free. Let $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{Q})$ be a filtered probability space (satisfying the usual conditions) such that for all $t \geq 0$ the σ-algebra $\mathcal{G}_t$ contains both, the market information up to time t and the information whether the default of the investor or its counterpart has occurred or not up to time t. $\mathbb{Q}$ is a (not necessarily unique) "spot martingale measure".

Modelling of default times

Let τ_0 denote the default time of the investor and τ_2 the default time of the counterparty. Suppose that the underlying financial market model is arbitrage-free. Let $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{Q})$ be a filtered probability space (satisfying the usual conditions) such that for all $t \geq 0$ the σ-algebra $\mathcal{G}_t$ contains both, the market information up to time t and the information whether the default of the investor or its counterpart has occurred or not up to time t. $\mathbb{Q}$ is a (not necessarily unique) "spot martingale measure".

Let $\tau := \min\{\tau_0, \tau_2\}$ (i. e., the "first-to-default time"). Both, τ_0 and τ_2 are $\mathbf{G}$-stopping times.

Modelling of default times

Let τ_0 denote the default time of the investor and τ_2 the default time of the counterparty. Suppose that the underlying financial market model is arbitrage-free. Let $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{Q})$ be a filtered probability space (satisfying the usual conditions) such that for all $t \geq 0$ the σ-algebra $\mathcal{G}_t$ contains both, the market information up to time t and the information whether the default of the investor or its counterpart has occurred or not up to time t. $\mathbb{Q}$ is a (not necessarily unique) "spot martingale measure".

Let $\tau := \min\{\tau_0, \tau_2\}$ (i. e., the "first-to-default time"). Both, τ_0 and τ_2 are $\mathbf{G}$-stopping times. Consider the $\mathbf{G}$-stopping time $\tau^* := \min\{\tau, T\}$. As we will see, a stochastic analysis of CCR builds on a consequent and repeated use of the positive functions $x^+ := \max\{x, 0\}$ and $x^- := x^+ - x = (-x)^+ = \max\{-x, 0\}$ $(x \in \mathbb{R})$.

All possible bilateral CCR scenarios

Let $k \in \{0, 2\}$. Consider the following sets:

All possible bilateral CCR scenarios

Let $k \in \{0, 2\}$. Consider the following sets:

- $N := \{\tau_0 > T \text{ and } \tau_2 > T\}$ (i. e., both, party 0 and party 2 do not default until T);

All possible bilateral CCR scenarios

UNIVERSITY OF Southampton School of Mathematics

Let $k \in \{0, 2\}$. Consider the following sets:

- $N := \{\tau_0 > T \text{ and } \tau_2 > T\}$ (i. e., both, party 0 and party 2 do not default until T);
- $A_k^- := \{\tau_k \leq T \text{ and } \tau_k < \tau_{2-k}\}$ (i. e., party k defaults first - until T);

All possible bilateral CCR scenarios

UNIVERSITY OF Southampton
School of Mathematics

Let $k \in \{0, 2\}$. Consider the following sets:

- $N := \{\tau_0 > T \text{ and } \tau_2 > T\}$ (i. e., both, party 0 and party 2 do not default until T);
- $A_k^- := \{\tau_k \leq T \text{ and } \tau_k < \tau_{2-k}\}$ (i. e., party k defaults first - until T);
- $A_k := \{\tau_k \leq T \text{ and } \tau_k = \tau_{2-k}\} = A_{2-k} =: A_{\text{sim}}$ (i. e., party 0 and party 2 default simultaneously - until T).

All possible bilateral CCR scenarios

Let $k \in \{0, 2\}$. Consider the following sets:

- $N := \{\tau_0 > T \text{ and } \tau_2 > T\}$ (i. e., both, party 0 and party 2 do not default until T);
- $A_k^- := \{\tau_k \leq T \text{ and } \tau_k < \tau_{2-k}\}$ (i. e., party k defaults first - until T);
- $A_k := \{\tau_k \leq T \text{ and } \tau_k = \tau_{2-k}\} = A_{2-k} =: A_{\text{sim}}$ (i. e., party 0 and party 2 default simultaneously - until T).

Observation

$$\Omega = N \,\dot\cup\, A_0^- \,\dot\cup\, A_2^- \,\dot\cup\, A_{sim} \,.$$

All possible bilateral CCR scenarios

Let $k \in \{0, 2\}$. Consider the following sets:

- $N := \{\tau_0 > T \text{ and } \tau_2 > T\}$ (i. e., both, party 0 and party 2 do not default until T);
- $A_k^- := \{\tau_k \leq T \text{ and } \tau_k < \tau_{2-k}\}$ (i. e., party k defaults first - until T);
- $A_k := \{\tau_k \leq T \text{ and } \tau_k = \tau_{2-k}\} = A_{2-k} =: A_{\text{sim}}$ (i. e., party 0 and party 2 default simultaneously - until T).

Observation

$$\Omega = N \,\dot\cup\, A_0^- \,\dot\cup\, A_2^- \,\dot\cup\, A_{sim}\,.$$

Moreover, $N \in \mathcal{G}_{\tau^*}$, $A_k^- \in \mathcal{G}_{\tau^*}$ *and hence also* $A_{sim} = \left(N \,\dot\cup\, A_0^- \,\dot\cup\, A_2^-\right)^c \in \mathcal{G}_{\tau^*}$.

Mark-to-Market value (MtM)

The trading book of a bank is based on the principle of “fair value accounting” (FAS 157 (US), respectively IAS 39 (EU)). It refers to accounting for the “fair value” of an asset or liability based on the current market price. Positions are “marked to market” on daily basis (via calibration of models to market data).

Mark-to-Market value (MtM)

UNIVERSITY OF Southampton
School of Mathematics

The trading book of a bank is based on the principle of "fair value accounting" (FAS 157 (US), respectively IAS 39 (EU)). It refers to accounting for the "fair value" of an asset or liability based on the current market price. Positions are "marked to market" on daily basis (via calibration of models to market data).

Deals are "fair" at the commencement of the financial transaction. I. e., their net present value at $t = 0$ equals zero: $M(0) := NPV(0) := 0$. However, as time goes by, the financial transaction is "marked to market", implying that at $0 < t \leq T$ $M(t) \equiv NPV(t) \neq 0$.

Mark-to-Market value (MtM)

The trading book of a bank is based on the principle of "fair value accounting" (FAS 157 (US), respectively IAS 39 (EU)). It refers to accounting for the "fair value" of an asset or liability based on the current market price. Positions are "marked to market" on daily basis (via calibration of models to market data).

Deals are "fair" at the commencement of the financial transaction. I. e., their net present value at $t = 0$ equals zero: $M(0) := NPV(0) := 0$. However, as time goes by, the financial transaction is "marked to market", implying that at $0 < t \leq T$ $M(t) \equiv NPV(t) \neq 0$.

Seen from $t = 0$, $M(t) : \Omega \to \mathbb{R}$ is a real-valued random variable (which can have a negative value if the trade is "out-of-the-money").

Money conservation property I

We follow a basic accounting property according to which a liability for party k represents an asset for k's counterparty $2-k$ itself (and conversely).

Money conservation property I

We follow a basic accounting property according to which a liability for party k represents an asset for k's counterparty $2-k$ itself (and conversely). An important special case of the basic accounting property is given by all processes satisfying the

Definition (Money Conservation Property)

Let $0 \leq t \leq T$ and $k \in \{0,2\}$. A cash flow $X = (X_t)_{0 \leq t \leq T}$ between two trading parties satisfies the Money Conservation Property (MCP) at t iff there exists $k \in \{0,2\}$ such that

$$X_t(k) = -X_t(2-k)\,.$$

Money conservation property I

Moreover, we follow a basic accounting property according to which a liability for party k represents an asset for k's counterparty $2-k$ itself (and conversely). An important special case of the basic accounting property is given by all processes satisfying the

Definition (Money Conservation Property)

Let $0 \leq t \leq T$ and $k \in \{0, 2\}$. A cash flow $X = (X_t)_{0 \leq t \leq T}$ between two trading parties satisfies the Money Conservation Property (MCP) at t iff there exists $k \in \{0, 2\}$ such that

$$X_t(k, 2-k) = -X_t(2-k, k) .$$

Money conservation property I

Moreover, we follow a basic accounting property according to which a liability for party k represents an asset for k's counterparty $2-k$ itself (and conversely). An important special case of the basic accounting property is given by all processes satisfying the

Definition (Money Conservation Property)

Let $0 \leq t \leq T$ and $k \in \{0,2\}$. A cash flow $X = (X_t)_{0 \leq t \leq T}$ between two trading parties satisfies the Money Conservation Property (MCP) at t iff there exists $k \in \{0,2\}$ such that

$$\left(X_t(k, 2-k) + X_t(2-k, k)\right)\Big|A = 0 \text{ for all } A \in \{A_0^-, A_2^-, A_{\text{sim}}, N\}.$$

Money conservation property II

In particular, (at t) such “MCP processes” X satisfy the following important property in relation to CCR:

$$\left(X_t(k, 2-k)\right)^+ = \left(X_t(2-k, k)\right)^- \ \forall k \in \{0, 2\}$$

Money conservation property II

In particular, (at t) such “MCP processes” X satisfy the following important property in relation to CCR:

$$\left(X_t(k, 2-k)\right)^+ = \left(X_t(2-k, k)\right)^- \ \forall k \in \{0, 2\}$$

Standing Assumption

Any “non-vulnerable” cash flow (i. e., any cash flow which is not accounting for CCR) is assumed to satisfy the MCP at any $t \in [0, T]$.

Money conservation property II

In particular, (at t) such "MCP processes" X satisfy the following important property in relation to CCR:

$$\left(X_t(k, 2-k)\right)^+ = \left(X_t(2-k, k)\right)^- \ \forall k \in \{0, 2\}$$

Standing Assumption

Any "non-vulnerable" cash flow (i. e., any cash flow which is not accounting for CCR) is assumed to satisfy the MCP at any $t \in [0, T]$.

Example

The following processes do not satisfy the MCP at t*:*

Money conservation property II

In particular, (at t) such "MCP processes" X satisfy the following important property in relation to CCR:

$$\big(X_t(k,2-k)\big)^+ = \big(X_t(2-k,k)\big)^- \quad \forall k \in \{0,2\}$$

Standing Assumption

Any "non-vulnerable" cash flow (i. e., any cash flow which is not accounting for CCR) is assumed to satisfy the MCP at any $t \in [0,T]$.

Example

The following processes do not satisfy the MCP at t*:*

(i) $Y_t(k,2-k)(\omega) := \mathbb{1}_{A_k^-}(\omega) + \mathbb{1}_{A_{2-k}^-}(\omega)\sin(t)LGD_k$;

Money conservation property II

In particular, (at t) such “MCP processes” X satisfy the following important property in relation to CCR:

$$\big(X_t(k, 2-k)\big)^+ = \big(X_t(2-k, k)\big)^- \quad \forall k \in \{0, 2\}$$

Standing Assumption

Any “non-vulnerable” cash flow (i. e., any cash flow which is not accounting for CCR) is assumed to satisfy the MCP at any $t \in [0, T]$.

Example

The following processes do not satisfy the MCP at t*:*

(i) $Y_t(k, 2-k)(\omega) := \mathbb{1}_{A_k^-}(\omega) + \mathbb{1}_{A_{2-k}^-}(\omega)\sin(t) LGD_k$;

(ii) $\Lambda_t(k, 2-k) := M_k(t) - LGD_{2-k}(M_k(t))^+$.

Representation of the MtM value M - FTAP I

Fix $k \in \{0, 2\}$ and let $0 \leq s \leq t \leq T$.

Representation of the MtM value M - FTAP I

Fix $k \in \{0, 2\}$ and let $0 \leq s \leq t \leq T$. Consider the random variable

$$\Pi_k^{(s,t]} := D(0,s)^{-1} \int_{(s,t]} D(0,u) d\Phi_k(u) \,,$$

where Φ_k (viewed from party k) denotes a non-vulnerable cumulative dividend process of the portfolio over the time horizon $[s,t]$ and $D(0,\cdot)$ a continuous $\mathbf{G}$-adapted discount factor process (which both are assumed to be of finite variation).

Representation of the MtM value M - FTAP I

Fix $k \in \{0,2\}$ and let $0 \leq s \leq t \leq T$. Consider the random variable

$$\Pi_k^{(s,t]} := D(0,s)^{-1} \int_{(s,t]} D(0,u) d\Phi_k(u) \, ,$$

where Φ_k (viewed from party k) denotes a non-vulnerable cumulative dividend process of the portfolio over the time horizon $[s,t]$ and $D(0,\cdot)$ a continuous $\mathbf{G}$-adapted discount factor process (which both are assumed to be of finite variation). $D(0,s)\Pi_k^{(s,t]} = \int_{(s,t]} D(0,u)d\Phi_k(u)$ represents the sum of all cumulatively discounted future cash flows of the portfolio between s and t not accounting for CCR (seen from the point of view of party k).

Representation of the MtM value M - FTAP I

Fix $k \in \{0, 2\}$ and let $0 \leq s \leq t \leq T$. Consider the random variable

$$\Pi_k^{(s,t]} := D(0,s)^{-1} \int_{(s,t]} D(0,u) d\Phi_k(u) \, ,$$

where Φ_k (viewed from party k) denotes a non-vulnerable cumulative dividend process of the portfolio over the time horizon $[s, t]$ and $D(0, \cdot)$ a continuous $\mathbf{G}$-adapted discount factor process (which both are assumed to be of finite variation). $D(0,s)\Pi_k^{(s,t]} = \int_{(s,t]} D(0,u) d\Phi_k(u)$ represents the sum of all cumulatively discounted future cash flows of the portfolio between s and t not accounting for CCR (seen from the point of view of party k). Hence, $\Pi_k^{(v,t]} = 0$ for all $v \in [t, T]$.

Representation of the MtM value M - FTAP I

Fix $k \in \{0, 2\}$ and let $0 \le s \le t \le T$. Consider the random variable

$$\Pi_k^{(s,t]} := D(0,s)^{-1} \int_{(s,t]} D(0,u) d\Phi_k(u) \,,$$

where Φ_k (viewed from party k) denotes a non-vulnerable cumulative dividend process of the portfolio over the time horizon $[s,t]$ and $D(0,\cdot)$ a continuous $\mathbf{G}$-adapted discount factor process (which both are assumed to be of finite variation). $D(0,s)\Pi_k^{(s,t]} = \int_{(s,t]} D(0,u) d\Phi_k(u)$ represents the sum of all cumulatively discounted future cash flows of the portfolio between s and t not accounting for CCR (seen from the point of view of party k). Hence, $\Pi_k^{(v,t]} = 0$ for all $v \in [t,T]$. Notice that $\Phi_k = -\Phi_{2-k}$ and hence $\Pi_k^{(s,t]} = -\Pi_{2-k}^{(s,t]}$ for all $0 \le s \le t \le T$ (due to the MCP).

Representation of the MtM M value - FTAP II

Assume throughout that our financial market model does not allow arbitrage and that each CCR free contingent claim between party k and party $2-k$ in the portfolio (or netting set) is attainable therein.

Representation of the MtM M value - FTAP II

Assume throughout that our financial market model does not allow arbitrage and that each CCR free contingent claim between party k and party $2-k$ in the portfolio (or netting set) is attainable therein. Thus, seen from party k's point of view the CCR free mark-to-market process
$M_k = (M_k(t))_{0 \leq t \leq T} \equiv (M_t(k))_{0 \leq t < T}$ is then given by

$$M_k(t) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} \middle| \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} D(t,u)d\Phi_k(u) \middle| \mathcal{G}_t\right] = -M_{2-k}(t)\,,$$

where $\mathbb{Q}$ is a "spot martingale measure" (due to the risk-neutral valuation formula). In the following we fix $\mathbb{Q}$ and occasionally omit its extra description in the notation of (conditional) expectation operators.

The main CCR building blocks

UNIVERSITY OF Southampton
School of Mathematics

$$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$$

Random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t – seen from k's point of view

The main CCR building blocks

$$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$$

Random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t – seen from k's point of view

$$M_k(t) = \mathbb{E}^{\mathbb{Q}}[\Pi_k^{(t,T]} | \mathcal{G}_t] = -M_{2-k}(t)$$

Random NPV (or MtM) of $\Pi_k^{(t,u]}$ – represented as conditional expectation w.r.t. $\mathbb{Q}$, given the "information" $\mathcal{G}_t$

The main CCR building blocks

$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$	Random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t – seen from k's point of view
$M_k(t) = \mathbb{E}^{\mathbb{Q}}[\Pi_k^{(t,T]} \mid \mathcal{G}_t] = -M_{2-k}(t)$	Random NPV (or MtM) of $\Pi_k^{(t,u]}$ – represented as conditional expectation w.r.t. $\mathbb{Q}$, given the "information" $\mathcal{G}_t$
$0 \leq R_k < 1$	k's (random) recovery rate; i. e., the portion of the payoff from the MtM paid by party k to party $2-k$ in case of k's default

The main CCR building blocks

$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$	Random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t – seen from k's point of view
$M_k(t) = \mathbb{E}^{\mathbb{Q}}[\Pi_k^{(t,T]} \mid \mathcal{G}_t]$ $= -M_{2-k}(t)$	Random NPV (or MtM) of $\Pi_k^{(t,u]}$ – represented as conditional expectation w.r.t. $\mathbb{Q}$, given the "information" $\mathcal{G}_t$
$0 \leq R_k < 1$	k's (random) recovery rate; i. e., the portion of the payoff from the MtM paid by party k to party $2-k$ in case of k's default
$0 < \mathrm{LGD}_k := 1 - R_k \leq 1$	k's (random) Loss Given Default

The main CCR building blocks

$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$	Random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t – seen from k's point of view
$M_k(t) = \mathbb{E}^{\mathbb{Q}}[\Pi_k^{(t,T]} \vert \mathcal{G}_t]$ $= -M_{2-k}(t)$	Random NPV (or MtM) of $\Pi_k^{(t,u]}$ – represented as conditional expectation w.r.t. $\mathbb{Q}$, given the "information" $\mathcal{G}_t$
$0 \leq R_k < 1$	k's (random) recovery rate; i. e., the portion of the payoff from the MtM paid by party k to party $2-k$ in case of k's default
$0 < \text{LGD}_k := 1 - R_k \leq 1$	k's (random) Loss Given Default
$D(t,u) := D(0,u)/D(0,t)$	discount factor at time t for time $u > t$ (can be random)

Valuation of defaultable claims I

Defaultable claims can be valued by interpreting them as portfolios of claims between non-defaultable counterparties including the riskless claim and mutual default protection contracts. Let $k \in \{0, 2\}$. Seen e. g. from the point of view of party k the latter says:

Party k sells to party $2 - k$ default protection on party $2 - k$ contingent to an amount specified by an ISDA close-out rule.

Valuation of defaultable claims I

Defaultable claims can be valued by interpreting them as portfolios of claims between non-defaultable counterparties including the riskless claim and mutual default protection contracts. Let $k \in \{0, 2\}$. Seen e. g. from the point of view of party k the latter says:

Party k sells to party $2-k$ default protection on party $2-k$ contingent to an amount specified by an ISDA close-out rule.

Let $t \in [0, T]$ such that $\rrbracket t, \tau^* \rrbracket \neq \emptyset$. Let

- $M_t(k)$ be the mark-to-market value to party k in case both, party k and party $2-k$ do not default until T;
- $CVA_t(k \,|\, 2-k)$ be the value of default protection that party k sells to party $2-k$ contingent on the default of party $2-k$.

Valuation of defaultable claims II

At t party k requires a payment of the “CCR risk premium” $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$.

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k.

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\mathsf{sim}} \neq \emptyset$ is not explicitly excluded!

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \mid 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \mid k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \mid 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \mid k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

$$k \text{ offers a payment of } M_t(k) \text{ to start a deal with } 2-k\,.$$

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \mid 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \mid k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

$$2-k \text{ rejects and requires } M_t(k) + CVA_t(2-k \mid k) \text{ instead} \quad .$$

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

$$-\Big(M_t(k) + CVA_t(2-k \,|\, k)\Big)\ ?$$

Valuation of defaultable claims II

At t party k requires a payment of the “CCR risk premium” $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the “bilaterally CCR-adjusted” value (defined as “fair value” in FAS 157):

k offers a payment of $M_t(k) + CVA_t(2-k \,|\, k) - CVA_t(k \,|\, 2-k)$, i. e.,

Valuation of defaultable claims II

At t party k requires a payment of the “CCR risk premium” $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the “bilaterally CCR-adjusted” value (defined as “fair value” in FAS 157):

k offers a payment of $M_t(k) - \big(CVA_t(k \,|\, 2-k) - CVA_t(2-k \,|\, k)\big) =: V_t(k)$.

Valuation of defaultable claims II

UNIVERSITY OF Southampton School of Mathematics

At t party k requires a payment of the “CCR risk premium” $CVA_t(k \,|\, 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the “bilaterally CCR-adjusted” value (defined as “fair value” in FAS 157):

$$-V_t(k) = -\Big(M_t(k) - \big(CVA_t(k \,|\, 2-k) - CVA_t(2-k \,|\, k)\big)\Big) \overset{(!)}{=} V_t(2-k)\,;$$

Valuation of defaultable claims II

At t party k requires a payment of the "CCR risk premium" $CVA_t(k \mid 2-k)$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \mid k)$ from party k to be compensated for the risk of a default of party k. Note that $A_{\text{sim}} \neq \emptyset$ is not explicitly excluded! Therefore, given the MCP party k reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

$$-V_t(k) = -\Big(M_t(k) - \big(CVA_t(k \mid 2-k) - CVA_t(2-k \mid k)\big)\Big) \overset{(!)}{=} V_t(2-k)\,;$$

Hence, the deal between k and $2-k$ is agreed!

CCR - Closing out practice I

Suppose, party k defaults (first).

CCR - Closing out practice I

Suppose, party k defaults (first).

The surviving party $2-k$ should then evaluate all terminated transactions to claim for at least a partial reimbursement (after the application of "netting rules") to fix the transactions (including the collateral).

CCR - Closing out practice I

Suppose, party k defaults (first).

The surviving party $2-k$ should then evaluate all terminated transactions to claim for at least a partial reimbursement (after the application of “netting rules”) to fix the transactions (including the collateral).

In order to maintain its market position, party k enters into a similar financial contract with another counterparty.

CCR - Closing out practice I

Suppose, party k defaults (first).

The surviving party $2-k$ should then evaluate all terminated transactions to claim for at least a partial reimbursement (after the application of “netting rules”) to fix the transactions (including the collateral).

In order to maintain its market position, party k enters into a similar financial contract with another counterparty.

Since the market position of k is unchanged after replacing the contract, the loss is determined by the contract’s “replacement value” at (or shortly after) τ_k.

CCR - Closing out practice I

Suppose, party k defaults (first).

The surviving party $2-k$ should then evaluate all terminated transactions to claim for at least a partial reimbursement (after the application of “netting rules”) to fix the transactions (including the collateral).

In order to maintain its market position, party k enters into a similar financial contract with another counterparty.

Since the market position of k is unchanged after replacing the contract, the loss is determined by the contract’s “replacement value” at (or shortly after) τ_k. In general, the partial reimbursement to party $2-k$ involves a payment of $(100\,\mathrm{LGD}_k)\%$ of the “replacement value” at (or shortly after) τ_k.

CCR - Closing out practice II

This process is known as “close-out”. The ISDA Master Agreement defines the term “close-out amount” *to be the amount of the losses or costs of the surviving party* $2-k$ *that would incur by replacement or provision of an economic equivalent*. To this end, ISDA introduced so called “close-out rules”.

CCR - Closing out practice II

This process is known as “close-out”. The ISDA Master Agreement defines the term “close-out amount” *to be the amount of the losses or costs of the surviving party* $2-k$ *that would incur by replacement or provision of an economic equivalent*. To this end, ISDA introduced so called “close-out rules”.

This leads us to the introduction of a useful definition which generalises the ISDA close-out approach (as we will see very soon).

CCR - Closing out practice III

Definition (Generalised close-out cash flow)

Let $k \in \{0, 2\}$ and $f, g : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Let H_k be the stochastic process (seen from the viewpoint of party k), defined as

$$\begin{aligned}
H_k(t) \quad := \quad & f\Big(\text{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+\Big)\mathbb{1}_{A^-_{2-k}} \\
- \quad & f\Big(\text{LGD}_k, (M_k(t))^-, (M_k(t))^+\Big)\mathbb{1}_{A^-_k} \\
+ \quad & g\Big(\text{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+\Big)\mathbb{1}_{A_{\text{sim}}} \\
- \quad & g\Big(\text{LGD}_k, (M_k(t))^-, (M_k(t))^+\Big)\mathbb{1}_{A_{\text{sim}}}
\end{aligned}$$

where $0 \leq t \leq T$. If $H_k(T) = 0$, then H_k is called a (symmetric) generalised close-out cash flow.

CCR - Closing out practice IV

Notice that - by construction - a generalised close-out cashflow already satisfies the MCP (due to its symmetry!) and $H_k \mathbb{1}_N \equiv 0$.

CCR - Closing out practice IV

Notice that - by construction - a generalised close-out cashflow already satisfies the MCP (due to its symmetry!) and $H_k \mathbb{1}_N \equiv 0$.
Further important pieces of notation:

CCR - Closing out practice IV

Notice that - by construction - a generalised close-out cashflow already satisfies the MCP (due to its symmetry!) and $H_k \mathbb{1}_N \equiv 0$.
Further important pieces of notation:

Let $\tau : \Omega \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ be an arbitrary random "time" (e. g., a stopping time) and $X : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$ a (real-valued) stochastic process. On $\{\tau < \infty\}$ the random variable $X_\tau : \Omega \longrightarrow \mathbb{R}$ is defined through

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) = X(\tau(\omega), \omega) \, .$$

CCR - Closing out practice IV

UNIVERSITY OF Southampton
School of Mathematics

Notice that - by construction - a generalised close-out cashflow already satisfies the MCP (due to its symmetry!) and $H_k 1\!\!1_N \equiv 0$.
Further important pieces of notation:

Let $\tau : \Omega \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ be an arbitrary random "time" (e. g., a stopping time) and $X : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$ a (real-valued) stochastic process. On $\{\tau < \infty\}$ the random variable $X_\tau : \Omega \longrightarrow \mathbb{R}$ is defined through

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) = X(\tau(\omega), \omega)\,.$$

Suppose that $\Omega \setminus N \neq \emptyset$. Choose $\omega \in \Omega \setminus N$. Given the - simplifying - assumption that there is no strictly positive margin period of risk, a non-zero close-out has to be settled at $\tau^*(\omega) = \min\{\tau_0(\omega), \tau_2(\omega), T\}$. For simplicity, let us also assume that no collateral is exchanged between party k and party $2 - k$ (until τ^*).

Bipartite ISDA CCR free close-out I

In the following we fix a very important building block which will appear repeatedly. It plays a fundamental role in ISDA's CCR free close-out rule. Let $k \in \{0, 2\}$ and $0 \leq t \leq T$. We put

Bipartite ISDA CCR free close-out I

In the following we fix a very important building block which will appear repeatedly. It plays a fundamental role in ISDA's CCR free close-out rule. Let $k \in \{0, 2\}$ and $0 \leq t \leq T$. We put

$$\begin{aligned}\Lambda_k(t) &:= M_k(t) - \text{LGD}_{2-k}(M_k(t))^+ \\ &\overset{(!)}{=} f^*(\text{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+)\end{aligned}$$

where $f^*(l, m_-, m_+) := m_-(1 - l) - m_+$ for all $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$

Bipartite ISDA CCR free close-out I

UNIVERSITY OF Southampton School of Mathematics

In the following we fix a very important building block which will appear repeatedly. It plays a fundamental role in ISDA's CCR free close-out rule. Let $k \in \{0, 2\}$ and $0 \leq t \leq T$. We put

$$\begin{aligned} \Lambda_k(t) &:= M_k(t) - \mathsf{LGD}_{2-k}(M_k(t))^+ \\ &\overset{(!)}{=} f^*(\mathsf{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+) \end{aligned}$$

where $f^*(l, m_-, m_+) := m_-(1-l) - m_+$ for all $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$ (why?).

Bipartite ISDA CCR free close-out I

In the following we fix a very important building block which will appear repeatedly. It plays a fundamental role in ISDA's CCR free close-out rule. Let $k \in \{0, 2\}$ and $0 \leq t \leq T$. We put

$$\begin{aligned} \Lambda_k(t) &:= M_k(t) - \mathrm{LGD}_{2-k}(M_k(t))^+ \\ &\overset{(!)}{=} f^*(\mathrm{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+) \end{aligned}$$

where $f^*(l, m_-, m_+) := m_-(1 - l) - m_+$ for all $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$ (why?). Notice that $\Lambda_k(T) \overset{(!)}{=} 0$.

Bipartite ISDA CCR free close-out II

University of Southampton School of Mathematics

Let $k \in \{0, 2\}$. Next, we will give a precise representation of the generalised close-out cash flow random variable $H_k(\tau^*)$ (seen from the viewpoint of party k), if both parties "symmetrically" apply the same ISDA close-out rule.

Bipartite ISDA CCR free close-out II

Let $k \in \{0,2\}$. Next, we will give a precise representation of the generalised close-out cash flow random variable $H_k(\tau^*)$ (seen from the viewpoint of party k), if both parties "symmetrically" apply the same ISDA close-out rule.

To this end, recall that for all $k \in \{0,2\}$ we always have

$$H_k(\tau^*) \stackrel{\checkmark}{=} H_k(\tau^*)\mathbb{1}_{A_0-} + H_k(\tau^*)\mathbb{1}_{A_2-} + H_k(\tau^*)\mathbb{1}_{A_{\text{sim}}} ,$$

since no non-zero close-out is required if both parties do not default strictly before T. In fact, by construction, we have $H_k(\tau^*)\mathbb{1}_N = H_k(T)\mathbb{1}_N = 0 \stackrel{\checkmark}{=} H_{2-k}(\tau^*)\mathbb{1}_N$ for all $k \in \{0,2\}$.

Bipartite ISDA CCR free close-out I

Fix $k \in \{0, 2\}$. Firstly, suppose $A^{-}_{2-k} \neq \emptyset$. Let $\omega \in A^{-}_{2-k}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

Bipartite ISDA CCR free close-out I

Fix $k \in \{0, 2\}$. Firstly, suppose $A^-_{2-k} \neq \emptyset$. Let $\omega \in A^-_{2-k}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

Let $0 \leq t \leq T$. Let $H_k(t) := H_t(k, 2-k)$ denote the random amount of the close-out cash flow at t seen from the viewpoint of party k. Since in any case $A^-_{2-k} \subseteq \{\tau_{2-k} = \tau^*\}$ (and $1\!\!1_\emptyset = 0$) the above table shows that in fact $H_k(\tau^*)\big|A^-_{2-k} = \Lambda_k(\tau^*)\big|A^-_{2-k}$, which is equivalent to:

Bipartite ISDA CCR free close-out I

Fix $k \in \{0,2\}$. Firstly, suppose $A^-_{2-k} \neq \emptyset$. Let $\omega \in A^-_{2-k}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

Let $0 \leq t \leq T$. Let $H_k(t) := H_t(k, 2-k)$ denote the random amount of the close-out cash flow at t seen from the viewpoint of party k. Since in any case $A^-_{2-k} \subseteq \{\tau_{2-k} = \tau^*\}$ (and $\mathbb{1}_\emptyset = 0$) the above table shows that in fact $H_k(\tau^*)\big|A^-_{2-k} = \Lambda_k(\tau^*)\big|A^-_{2-k}$, which is equivalent to:

$$\mathbb{1}_{A^-_{2-k}} H_k(\tau^*) = \mathbb{1}_{A^-_{2-k}} \left(R_{2-k}(M_k(\tau^*))^+ - (-M_k(\tau^*))^+\right)$$

Bipartite ISDA CCR free close-out I

Fix $k \in \{0, 2\}$. Firstly, suppose $A^-_{2-k} \neq \emptyset$. Let $\omega \in A^-_{2-k}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

Let $0 \leq t \leq T$. Let $H_k(t) := H_t(k, 2-k)$ denote the random amount of the close-out cash flow at t seen from the viewpoint of party k. Since in any case $A^-_{2-k} \subseteq \{\tau_{2-k} = \tau^*\}$ (and $1\!\!1_\emptyset = 0$) the above table shows that in fact $H_k(\tau^*)\big|A^-_{2-k} = \Lambda_k(\tau^*)\big|A^-_{2-k}$, which is equivalent to:

$$1\!\!1_{A^-_{2-k}} H_k(\tau^*) = 1\!\!1_{A^-_{2-k}} \left(R_{2-k}(M_k(\tau^*))^+ - (-M_k(\tau^*))^+\right) \overset{\checkmark}{=} 1\!\!1_{A^-_{2-k}} \Lambda_k(\tau^*) \,.$$

Bipartite ISDA CCR free close-out II

Fix $k \in \{0, 2\}$ and suppose now that $A_k^- \neq \emptyset$. Let $\omega \in A_k^-$. Seen from the viewpoint of party $2-k$ the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_{2-k}(\tau_k)(\omega) > 0$	$M_{2-k}(\tau_k)(\omega) \leq 0$
Party $2-k$ receives from party k	$R_k(\omega) \cdot M_{2-k}(\tau_k)(\omega)$	0
Party $2-k$ pays to party k	0	$-M_{2-k}(\tau_k)(\omega)$

Hence,

$$\mathbb{1}_{A_k^-} H_{2-k}(\tau^*) = \mathbb{1}_{A_k^-} \left(R_k(M_{2-k}(\tau^*))^+ - (-M_{2-k}(\tau^*))^+ \right) = \mathbb{1}_{A_k^-} \Lambda_{2-k}(\tau^*) \,.$$

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

Observation

Fix $k \in \{0,2\}$. Firstly let us assume that there are no simultaneous defaults $(i.\,e., A_{sim} = \emptyset)$. Then the ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $F_k(\tau^*)$, where

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

Observation

Fix $k \in \{0, 2\}$. Firstly let us assume that there are no simultaneous defaults ($i.\ e.$, $A_{sim} = \emptyset$). Then the ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $F_k(\tau^*)$, where

$$F_k(\tau^*) \quad := \quad \mathbb{1}_{A^-_{2-k}} \Lambda_k(\tau^*) - \mathbb{1}_{A^-_k} \Lambda_{2-k}(\tau^*)$$

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

Observation

Fix $k \in \{0,2\}$. Firstly let us assume that there are no simultaneous defaults $($i. e., $A_{sim} = \emptyset)$. Then the ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $F_k(\tau^*)$, where

$$
\begin{aligned}
F_k(\tau^*) \quad &:= \quad \mathbb{1}_{A^-_{2-k}} \Lambda_k(\tau^*) - \mathbb{1}_{A^-_k} \Lambda_{2-k}(\tau^*) \\
&= \quad \mathbb{1}_{A^-_k} \mathit{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A^-_{2-k}} \mathit{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\quad\quad - \mathbb{1}_{A^-_k} M_{2-k}(\tau^*) + \mathbb{1}_{A^-_{2-k}} M_k(\tau^*)
\end{aligned}
$$

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

Observation

Fix $k \in \{0,2\}$. Firstly let us assume that there are no simultaneous defaults $($i. e., $A_{sim} = \emptyset)$. Then the ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $F_k(\tau^*)$, where

$$
\begin{aligned}
F_k(\tau^*) \quad &:= \quad \mathbb{1}_{A_{2-k}^-}\Lambda_k(\tau^*) - \mathbb{1}_{A_k^-}\Lambda_{2-k}(\tau^*) \\
&= \quad \mathbb{1}_{A_k^-}\mathit{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-}\mathit{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\qquad - \mathbb{1}_{A_k^-}M_{2-k}(\tau^*) + \mathbb{1}_{A_{2-k}^-}M_k(\tau^*) \\
&\overset{(MCP)}{=} \quad \mathbb{1}_{A_k^-}\mathit{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-}\mathit{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\qquad + (\mathbb{1}_{A_0^- \cup A_2^-})M_k(\tau^*).
\end{aligned}
$$

Bipartite ISDA CCR free close-out IV

Fix $k \in \{0, 2\}$. The case of simultaneous defaults – seen from the point of view of party k - can be treated in the same way. Hence, by another application of the MCP we arrive at the following ISDA CCR free close-out cash flow random variable at τ^* (coinciding with the fourth term in formula (5) of Gregory's paper [6]):

Bipartite ISDA CCR free close-out IV

Fix $k \in \{0, 2\}$. The case of simultaneous defaults – seen from the point of view of party k - can be treated in the same way. Hence, by another application of the MCP we arrive at the following ISDA CCR free close-out cash flow random variable at τ^* (coinciding with the fourth term in formula (5) of Gregory's paper [6]):

$$G_k(\tau^*) \quad := \quad 1\!\!1_{A_{\mathrm{sim}}} \Lambda_k(\tau^*) - 1\!\!1_{A_{\mathrm{sim}}} \Lambda_{2-k}(\tau^*)$$

Bipartite ISDA CCR free close-out IV

Fix $k \in \{0, 2\}$. The case of simultaneous defaults – seen from the point of view of party k - can be treated in the same way. Hence, by another application of the MCP we arrive at the following ISDA CCR free close-out cash flow random variable at τ^* (coinciding with the fourth term in formula (5) of Gregory's paper [6]):

$$
\begin{aligned}
G_k(\tau^*) \quad &:= \quad \mathbb{1}_{A_{\text{sim}}} \Lambda_k(\tau^*) - \mathbb{1}_{A_{\text{sim}}} \Lambda_{2-k}(\tau^*) \\
&\overset{(MCP)}{=} \quad \mathbb{1}_{A_{\text{sim}}} \text{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{\text{sim}}} \text{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\qquad + \mathbb{1}_{A_{\text{sim}}} 2M_k(\tau^*).
\end{aligned}
$$

Bipartite ISDA CCR free close-out IV

Fix $k \in \{0, 2\}$. The case of simultaneous defaults – seen from the point of view of party k - can be treated in the same way. Hence, by another application of the MCP we arrive at the following ISDA CCR free close-out cash flow random variable at τ^* (coinciding with the fourth term in formula (5) of Gregory's paper [6]):

$$
\begin{aligned}
G_k(\tau^*) \quad &:= \quad \mathbb{1}_{A_{\text{sim}}} \Lambda_k(\tau^*) - \mathbb{1}_{A_{\text{sim}}} \Lambda_{2-k}(\tau^*) \\
&\overset{(MCP)}{=} \quad \mathbb{1}_{A_{\text{sim}}} \text{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{\text{sim}}} \text{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\quad + \mathbb{1}_{A_{\text{sim}}} 2M_k(\tau^*).
\end{aligned}
$$

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

$$
\begin{aligned}
H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\
&= \mathbb{1}_{A_k^-} \mathsf{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-} \mathsf{LGD}_{2-k}(M_k(\tau^*))^+ \\
&\quad + (\mathbb{1}_{A_0^- \cup A_2^-}) M_k(\tau^*) \\
&\quad + \mathbb{1}_{A_{\text{sim}}} \big(2M_k(\tau^*) - \mathsf{LGD}_{2-k}(M_k(\tau^*))^+ + \mathsf{LGD}_k(M_{2-k}(\tau^*))^+\big)
\end{aligned}
$$

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

$$
\begin{aligned}
H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\
&= \mathbb{1}_{A_k^- \cup A_{\mathrm{sim}}} \mathrm{LGD}_k (M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^- \cup A_{\mathrm{sim}}} \mathrm{LGD}_{2-k} (M_k(\tau^*))^+ \\
&\quad + (\mathbb{1}_{A_0^- \cup A_2^-} + 2\mathbb{1}_{A_{\mathrm{sim}}}) M_k(\tau^*) \,.
\end{aligned}
$$

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0,2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

$$\begin{aligned} H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\ &= \mathbb{1}_{A_k^- \dot\cup A_{\text{sim}}} \mathrm{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^- \dot\cup A_{\text{sim}}} \mathrm{LGD}_{2-k}(M_k(\tau^*))^+ \\ &\quad + (1 - \mathbb{1}_N)\, M_k(\tau^*) + \mathbb{1}_{A_{\text{sim}}}\, M_k(\tau^*)\,. \end{aligned}$$

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

$$
\begin{aligned}
H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\
&= \mathbb{1}_{A_k^- \cup A_{\text{sim}}} \mathrm{LGD}_k (M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^- \cup A_{\text{sim}}} \mathrm{LGD}_{2-k} (M_k(\tau^*))^+ \\
&\quad + M_k(\tau^*) + \mathbb{1}_{A_{\text{sim}}} \left((M_k(\tau^*))^+ - (M_k(\tau^*))^- \right).
\end{aligned}
$$

Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

By considering all possible cases (including the possibility of simultaneous defaults) a remaining simple bit of algebra therefore gives us - step by step - the important general representation of $H_k^*(\tau^*)$:

$$\begin{aligned} H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\ &\overset{(!)}{=} \mathbb{1}_{A_k^- \cup A_{\text{sim}}} \text{LGD}_k (M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^- \cup A_{\text{sim}}} \text{LGD}_{2-k}(M_k(\tau^*))^+ \\ &\quad + M_k(\tau^*) + \mathbb{1}_{A_{\text{sim}}} \left((M_k(\tau^*))^+ - \boxed{(M_{2-k}(\tau^*))^+} \right) . \end{aligned}$$

Bipartite ISDA CCR free close-out VI

UNIVERSITY OF Southampton School of Mathematics

Thus,

Bipartite ISDA CCR free close-out VI

Thus,

Proposition

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$*. Seen from the viewpoint of party* k*, the random variable* $H_k^*(\tau^*)$ *is given as*

$$\begin{aligned} H_k^*(\tau^*) &= M_k(\tau^*) - B_{\tau^*}(k, 2-k) \\ &= M_k(\tau^*) - \left(X_{2-k}(\tau^*) - X_k(\tau^*)\right), \end{aligned}$$

where $B_{\tau^*}(k, 2-k) = X_{2-k}(\tau^*) - X_k(\tau^*) \overset{\checkmark}{=} -B_{\tau^*}(2-k, k)$ *and*

$$X_k(t) := \left(\mathbb{1}_{A_k^- \dot\cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)\left(M_{2-k}(t)\right)^+.$$

Bipartite ISDA CCR free close-out VI

Thus,

Proposition

Let $k \in \{0,2\}$ and $t \in [0,T]$. Seen from the viewpoint of party k, the random variable $H_k^*(\tau^*)$ is given as

$$\begin{aligned} H_k^*(\tau^*) &= M_k(\tau^*) - B_{\tau^*}(k, 2-k) \\ &= M_k(\tau^*) - \left(X_{2-k}(\tau^*) - X_k(\tau^*)\right), \end{aligned}$$

where $B_{\tau^*}(k,2-k) = X_{2-k}(\tau^*) - X_k(\tau^*) \overset{\checkmark}{=} -B_{\tau^*}(2-k,k)$ and

$$X_k(t) := \left(\mathbb{1}_{A_k^- \dot\cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(t))^+.$$

Actually, we have seen more. Namely,

Bipartite ISDA CCR free close-out VII

University of Southampton
School of Mathematics

Proposition

Let $k \in \{0, 2\}$ and $t \in [0, T]$. Seen from the viewpoint of party k, the random variable $H_k^*(t)$ is given as

$$\begin{aligned} H_k^*(t) &= (1 - \mathbb{1}_N) M_k(t) - B_t(k, 2-k) \\ &= (1 - \mathbb{1}_N) M_k(t) - \big(X_{2-k}(t) - X_k(t) \big) , \end{aligned}$$

Bipartite ISDA CCR free close-out VII

Proposition

Let $k \in \{0, 2\}$ and $t \in [0, T]$. Seen from the viewpoint of party k, the random variable $H_k^*(t)$ is given as

$$\begin{aligned} H_k^*(t) &= (1 - \mathbb{1}_N)M_k(t) - B_t(k, 2-k) \\ &= (1 - \mathbb{1}_N)M_k(t) - \big(X_{2-k}(t) - X_k(t)\big)\,, \end{aligned}$$

where $B_t(k, 2-k) := X_{2-k}(t) - X_k(t) \overset{\checkmark}{=} -B_t(2-k, k)$ *and*

$$X_k(t) := \big(\mathbb{1}_{A_k^- \dot\cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\big)(M_{2-k}(t))^+ .$$

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Observation (ISDA CCR free restrictions)

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Then*

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Observation (ISDA CCR free restrictions)

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Then*

(i) $B_t(k, 2-k)\mathbb{1}_{A_k^-} = -LGD_k(M_{2-k}(t))^+\mathbb{1}_{A_k^-}$

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Observation (ISDA CCR free restrictions)

Let $k \in \{0,2\}$ *and* $t \in [0,T]$. *Then*

(i) $B_t(k,2-k)\mathbb{1}_{A_k^-} = -LGD_k(M_{2-k}(t))^+\mathbb{1}_{A_k^-}$

(ii) $B_t(k,2-k)\mathbb{1}_{A_{sim}} = \big(R_k(M_{2-k}(t))^+ - R_{2-k}(M_k(t))^+\big)\mathbb{1}_{A_{sim}}$

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Observation (ISDA CCR free restrictions)

Let $k \in \{0,2\}$ *and* $t \in [0,T]$*. Then*

(i) $B_t(k, 2-k)\mathbb{1}_{A_k^-} = -LGD_k(M_{2-k}(t))^+\mathbb{1}_{A_k^-}$

(ii) $B_t(k, 2-k)\mathbb{1}_{A_{sim}} = \big(R_k(M_{2-k}(t))^+ - R_{2-k}(M_k(t))^+\big)\mathbb{1}_{A_{sim}}$

(iii) $B_t(k, 2-k)\mathbb{1}_N = 0.$

Bipartite ISDA CCR free close-out VIII

Having an axiomatic approach in our sight, let us explicitly write down the following

Observation (ISDA CCR free restrictions)

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Then*

(i) $B_t(k, 2-k)\mathbb{1}_{A_k^-} = -LGD_k(M_{2-k}(t))^+\mathbb{1}_{A_k^-}$

(ii) $B_t(k, 2-k)\mathbb{1}_{A_{sim}} = \big(R_k(M_{2-k}(t))^+ - R_{2-k}(M_k(t))^+\big)\mathbb{1}_{A_{sim}}$

(iii) $B_t(k, 2-k)\mathbb{1}_N = 0.$

The ISDA CCR free restrictions simply say that both parties, party 0 and party 2 close-out their positions according to the lines of the bipartite ISDA CCR free close-out rule by taking into account precisely all - possible - default scenarios.

Bipartite ISDA CCR free close-out IX

UNIVERSITY OF Southampton School of Mathematics

Definition

Let $k \in \{0, 2\}, t \in [0, T]$ and

$$f^*(l, m_-, m_+) := m_-(1 - l) - m_+ \, ,$$

where $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

Bipartite ISDA CCR free close-out IX

Definition

Let $k \in \{0, 2\}, t \in [0, T]$ and

$$f^*(l, m_-, m_+) := m_-(1 - l) - m_+ ,$$

where $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$. Then the stochastic process H^*, defined via

$$\begin{aligned} H^*_k(\omega, t) \quad &:= \quad \mathbb{1}_{A^-_{2-k} \cup A_{\mathsf{sim}}}(\omega) f^*\big(\mathsf{LGD}_{2-k}, (M_{2-k}(\omega, t))^-, (M_{2-k}(\omega, t))^+\big) \\ &- \quad \mathbb{1}_{A^-_k \cup A_{\mathsf{sim}}}(\omega) f^*\big(\mathsf{LGD}_k, (M_k(\omega, t))^-, (M_k(\omega, t))^+\big) \end{aligned}$$

is called a symmetrically bipartite ISDA CCR free close-out cash flow (seen from the viewpoint of party k).

Bipartite ISDA CCR free close-out IX

UNIVERSITY OF Southampton School of Mathematics

Definition

Let $k \in \{0, 2\}$, $t \in [0, T]$ and

$$f^*(l, m_-, m_+) := m_-(1 - l) - m_+ ,$$

where $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$. Then the stochastic process H^*, defined via

$$\begin{aligned} H_k^*(\omega, t) \quad &:= \quad \mathbb{1}_{A_{2-k}^- \cup A_{\mathsf{sim}}}(\omega) f^*\left(\mathsf{LGD}_{2-k}, (M_{2-k}(\omega, t))^-, (M_{2-k}(\omega, t))^+\right) \\ &- \quad \mathbb{1}_{A_k^- \cup A_{\mathsf{sim}}}(\omega) f^*\left(\mathsf{LGD}_k, (M_k(\omega, t))^-, (M_k(\omega, t))^+\right) \end{aligned}$$

is called a symmetrically bipartite ISDA CCR free close-out cash flow (seen from the viewpoint of party k).

Hence, H^* is a special case of a generalised close-out cash flow in our sense.

Vulnerable cash flows I

Fix $k \in \{0, 2\}$ and $t \in [0, T]$. Let $t \leq \tau^*(\omega)$. Based on our previous representation of $H_k^*(\tau^*)(\omega)$ discounting to t immediately implies

Vulnerable cash flows I

Fix $k \in \{0, 2\}$ and $t \in [0, T]$. Let $t \leq \tau^*(\omega)$. Based on our previous representation of $H_k^*(\tau^*)(\omega)$ discounting to t immediately implies

Theorem and Definition

Let $t \in [0, T]$. On $\{t \leq \tau^*\}$ the t-discounted bipartite ISDA CCR free close-out cash flow amount seen from the point of view of party k at t is given by $D(t, \tau^*)H_k^*(\tau^*)$.

Fix $k \in \{0, 2\}$ and $t \in [0, T]$. Let $t \leq \tau^*(\omega)$. Based on our previous representation of $H_k^*(\tau^*)(\omega)$ discounting to t immediately implies

Theorem and Definition

Let $t \in [0, T]$. On $\{t \leq \tau^*\}$ the t-discounted bipartite ISDA CCR free close-out cash flow amount seen from the point of view of party k at t is given by $D(t, \tau^*)H_k^*(\tau^*)$. I. e.,

$$D(t, \tau^*)H_k^*(\tau^*) = D(t, \tau^*)\, M_k(\tau^*) - D(t, \tau^*)B_{\tau^*}(k, 2 - k)$$

on $\{t \leq \tau^*\}$,

Vulnerable cash flows I

Fix $k \in \{0, 2\}$ and $t \in [0, T]$. Let $t \le \tau^*(\omega)$. Based on our previous representation of $H_k^*(\tau^*)(\omega)$ discounting to t immediately implies

Theorem and Definition

Let $t \in [0, T]$. On $\{t \le \tau^*\}$ the t-discounted bipartite ISDA CCR free close-out cash flow amount seen from the point of view of party k at t is given by $D(t, \tau^*)H_k^*(\tau^*)$. I. e.,

$$D(t, \tau^*)H_k^*(\tau^*) = D(t, \tau^*)\, M_k(\tau^*) - D(t, \tau^*)B_{\tau^*}(k, 2-k)$$

on $\{t \le \tau^*\}$, where (as before) $B_{\tau^*}(k, 2-k) = X_{2-k}(\tau^*) - X_k(\tau^*)$ and

$$X_k(\tau^*) = \left(\mathbb{1}_{A_k^- \cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(\tau^*))^+ .$$

Vulnerable cash flows II

Fix $k \in \{0, 2\}$. Let $0 \leq t \leq u \leq T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t, u]$, discounted to time t (seen from k's point of view).

Vulnerable cash flows II

Fix $k \in \{0, 2\}$. Let $0 \le t \le u \le T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t, u]$, discounted to time t (seen from k's point of view). Let us similarly denote by $\widehat{\Pi}_k^{(t,T]}$ the random cumulative cash flow from the claim in $(t, T]$, discounted to time t (seen from k's point of view), yet accounting for CCR now.

Vulnerable cash flows II

Fix $k \in \{0, 2\}$. Let $0 \leq t \leq u \leq T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t, u]$, discounted to time t (seen from k's point of view). Let us similarly denote by $\widehat{\Pi}_k^{(t,T]}$ the random cumulative cash flow from the claim in $(t, T]$, discounted to time t (seen from k's point of view), yet accounting for CCR now.

By construction $\widehat{\Pi}_k^{(t,T]}$ should include both first-to-default scenarios *and* the scenario of a simultaneous default of both parties. To derive the structure of $\widehat{\Pi}_k^{(t,T]}$, we again assume that the MCP holds, as well as our basic accounting principle, and that each party applies the bipartite ISDA CCR free close-out rule.

Vulnerable cash flows II

Fix $k \in \{0, 2\}$. Let $0 \leq t \leq u \leq T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t, u]$, discounted to time t (seen from k's point of view). Let us similarly denote by $\widehat{\Pi}_k^{(t,T]}$ the random cumulative cash flow from the claim in $(t, T]$, discounted to time t (seen from k's point of view), yet accounting for CCR now.

By construction $\widehat{\Pi}_k^{(t,T]}$ should include both first-to-default scenarios *and* the scenario of a simultaneous default of both parties. To derive the structure of $\widehat{\Pi}_k^{(t,T]}$, we again assume that the MCP holds, as well as our basic accounting principle, and that each party applies the bipartite ISDA CCR free close-out rule.

Hence, on $\{t \leq \tau^*\}$ we put

$$\widehat{\Pi}_k^{(t,T]} := \Pi_k^{(t,\tau^*]} + D(t, \tau^*) H_k^*(\tau^*)$$

Vulnerable cash flows III

Recall that we also assume that the No-Arbitrage Principle is satisfied. Hence, under inclusion of all those listed assumptions, we obtain the following

Vulnerable cash flows III

Recall that we also assume that the No-Arbitrage Principle is satisfied. Hence, under inclusion of all those listed assumptions, we obtain the following

Lemma (Representation of $\widehat{\Pi}_k^{(t,T]}$)

Let $k \in \{0, 2\}$ and $t \in [0, T]$. On $\{t \leq \tau^*\}$, the random variable $\widehat{\Pi}_k^{(t,T]}$ can be written as

Vulnerable cash flows III

Recall that we also assume that the No-Arbitrage Principle is satisfied. Hence, under inclusion of all those listed assumptions, we obtain the following

Lemma (Representation of $\widehat{\Pi}_k^{(t,T]}$)

Let $k \in \{0,2\}$ *and* $t \in [0,T]$*. On* $\{t \leq \tau^*\}$*, the random variable* $\widehat{\Pi}_k^{(t,T]}$ *can be written as*

$$\widehat{\Pi}_k^{(t,T]} = \Pi_k^{(t,T]} - D(t,\tau^*)B_{\tau^*}(k,2-k) + D(t,\tau^*)\left(M_k(\tau^*) - \Pi_k^{(\tau^*,T]}\right).$$

Vulnerable cash flows III

Recall that we also assume that the No-Arbitrage Principle is satisfied. Hence, under inclusion of all those listed assumptions, we obtain the following

Lemma (Representation of $\widehat{\Pi}_k^{(t,T]}$)

Let $k \in \{0,2\}$ *and* $t \in [0,T]$*. On* $\{t \leq \tau^*\}$*, the random variable* $\widehat{\Pi}_k^{(t,T]}$ *can be written as*

$$\widehat{\Pi}_k^{(t,T]} = \Pi_k^{(t,T]} - D(t,\tau^*)B_{\tau^*}(k,2-k) + D(t,\tau^*)\Big(M_k(\tau^*) - \Pi_k^{(\tau^*,T]}\Big)\,.$$

In particular, we have $\widehat{\Pi}_k^{(t,T]} \stackrel{(MCP)}{=} -\widehat{\Pi}_{2-k}^{(t,T]}$.

Hence, a consequent application of the conditional expectation $\mathbb{E}^{\mathbb{Q}}\big[\,\cdot\,\big|\,\mathcal{G}_{\tau^*}\big]$ to $\widehat{\Pi}_k^{(t,T]}$, together with some stochastic analysis lead to the following crucial

Vulnerable cash flows IV

Theorem (Market prices of total bipartite CCR)

Let $k \in \{0, 2\}$ and $t \in [0, T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] =$$

Vulnerable cash flows IV

Theorem (Market prices of total bipartite CCR)

Let $k \in \{0, 2\}$ and $t \in [0, T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k)$$

Vulnerable cash flows IV

Theorem (Market prices of total bipartite CCR)

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] &= M_t(k) \\ &+ \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau^*)\left(\mathbb{1}_{A_k^- \cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(\tau^*))^+ \Big| \mathcal{G}_t\right] \end{aligned}$$

Vulnerable cash flows IV

Theorem (Market prices of total bipartite CCR)

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] &= M_t(k) \\
&+ \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau^*)\left(\mathbb{1}_{A_k^- \cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right] \\
&- \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau^*)\left(\mathbb{1}_{A_{2-k}^- \cup A_{sim}} LGD_{2-k} - \mathbb{1}_{A_{sim}}\right)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right].
\end{aligned}$$

Vulnerable cash flows V

Latter result contains an important and well-known special case. Namely the first appearance of “FTDCVA” and “FTDDVA”:

Vulnerable cash flows V

Latter result contains an important and well-known special case. Namely the first appearance of “FTDCVA” and “FTDDVA”:

Corollary (Brigo-Capponi (2009))

Let $k \in \{0,2\}$, $t \in [0,T]$ and assume that each party applies the bipartite ISDA CCR free close-out rule. Further assume that both, the No-Arbitrage Principle and the MCP are satisfied and that the basic accounting rule holds. If there are no simultaneous defaults (i. e., if $A_{sim} = \emptyset \quad \mathbb{Q}$-a. s.) then on $\{t \le \tau^*\}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] &\overset{(!)}{=} M_t(k) \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} \mathit{LGD}_k D(t,\tau^*)\left(M_{2-k}(\tau^*)\right)^+ \middle| \mathcal{G}_t\right] \\
&- \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{2-k}^-} \mathit{LGD}_{2-k} D(t,\tau^*)\left(M_k(\tau^*)\right)^+ \middle| \mathcal{G}_t\right].
\end{aligned}
$$

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$\mathrm{TCVA}_t(k|2-k)$

TCVA, TDVA and TBVA I

Definition

Let $k=0$ or $k=2$, $t\in[0,T]$ and $t\le\tau^*$. Put

$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k}\cup A_{\text{sim}}}\text{LGD}_{2-k}-\mathbb{1}_{A_{\text{sim}}}\right)D(t,\tau^*)(M_k(\tau^*))^+\middle|\mathcal{G}_t\right],$$

UNIVERSITY OF Southampton School of Mathematics

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k} \cup A_{\text{sim}}} \text{LGD}_{2-k} - \mathbb{1}_{A_{\text{sim}}}\right) D(t, \tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\text{TBVA}_t(k, 2-k) := \text{TCVA}_t(k|2-k) - \text{TCVA}_t(2-k|k),$$

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k}\cup A_{\text{sim}}}\text{LGD}_{2-k} - \mathbb{1}_{A_{\text{sim}}}\right) D(t,\tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\text{TBVA}_t(k, 2-k) := \text{TCVA}_t(k|2-k) - \text{TCVA}_t(2-k|k),$$

and...

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\mathsf{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k} \cup A_{\mathsf{sim}}} \mathsf{LGD}_{2-k} - \mathbb{1}_{A_{\mathsf{sim}}}\right) D(t, \tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\mathsf{TBVA}_t(k, 2-k) := \mathsf{TCVA}_t(k|2-k) - \mathsf{TCVA}_t(2-k|k),$$

and...

$$\mathsf{TDVA}_t(k, 2-k)$$

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k} \cup A_{\text{sim}}} \text{LGD}_{2-k} - \mathbb{1}_{A_{\text{sim}}}\right) D(t, \tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\text{TBVA}_t(k, 2-k) := \text{TCVA}_t(k|2-k) - \text{TCVA}_t(2-k|k),$$

and...

$$\text{TDVA}_t(k, 2-k) :=$$

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A_{2-k}^- \cup A_{\text{sim}}}\text{LGD}_{2-k} - \mathbb{1}_{A_{\text{sim}}}\right) D(t,\tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\text{TBVA}_t(k, 2-k) := \text{TCVA}_t(k|2-k) - \text{TCVA}_t(2-k|k),$$

and...

$$\text{TDVA}_t(k, 2-k) := \text{TCVA}_t(2-k|k)$$

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\mathsf{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k} \cup A_{\mathsf{sim}}} \mathsf{LGD}_{2-k} - \mathbb{1}_{A_{\mathsf{sim}}}\right) D(t, \tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\mathsf{TBVA}_t(k, 2-k) := \mathsf{TCVA}_t(k|2-k) - \mathsf{TCVA}_t(2-k|k),$$

and...

$$\mathsf{TDVA}_t(k, 2-k) := \mathsf{TCVA}_t(2-k|k) \rightsquigarrow (??)$$

TCVA, TDVA and TBVA I

Definition

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

$$\mathsf{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A_{2-k}^- \cup A_{\mathsf{sim}}} \mathsf{LGD}_{2-k} - \mathbb{1}_{A_{\mathsf{sim}}}\right) D(t,\tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

$$\mathsf{TBVA}_t(k, 2-k) := \mathsf{TCVA}_t(k|2-k) - \mathsf{TCVA}_t(2-k|k),$$

and...

$$\begin{aligned}\mathsf{TDVA}_t(k, 2-k) &:= \mathsf{TCVA}_t(2-k|k) \rightsquigarrow (??) \\ &:= \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A_k^- \cup A_{\mathsf{sim}}} \mathsf{LGD}_k - \mathbb{1}_{A_{\mathsf{sim}}}\right) D(t,\tau^*)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right].\end{aligned}$$

TCVA, TDVA and TBVA II

Definition (ctd.)

(i) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TCVA}_t(k \mid 2-k)$ is called Total Credit Valuation Adjustment at t, seen from the viewpoint of party k.

TCVA, TDVA and TBVA II

Definition (ctd.)

(i) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TCVA}_t(k \mid 2-k)$ is called Total Credit Valuation Adjustment at t, seen from the viewpoint of party k.

(ii) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TBVA}_t(k, 2-k)$ is called Total Bilateral Valuation Adjustment at t, seen from the viewpoint of party k.

TCVA, TDVA and TBVA II

Definition (ctd.)

(i) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TCVA}_t(k \,|\, 2-k)$ is called Total Credit Valuation Adjustment at t, seen from the viewpoint of party k.

(ii) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TBVA}_t(k, 2-k)$ is called Total Bilateral Valuation Adjustment at t, seen from the viewpoint of party k.

(iii) The real-valued $\mathcal{G}_t$-measurable random variable $\text{TDVA}_t(k, 2-k)$ is called Total Debit Valuation Adjustment at t, seen from the viewpoint of party k.

TCVA, TDVA and TBVA II

Definition (ctd.)

(i) The real-valued $\mathcal{G}_t$-measurable random variable $\mathrm{TCVA}_t(k \mid 2-k)$ is called Total Credit Valuation Adjustment at t, seen from the viewpoint of party k.

(ii) The real-valued $\mathcal{G}_t$-measurable random variable $\mathrm{TBVA}_t(k, 2-k)$ is called Total Bilateral Valuation Adjustment at t, seen from the viewpoint of party k.

(iii) The real-valued $\mathcal{G}_t$-measurable random variable $\mathrm{TDVA}_t(k, 2-k)$ is called Total Debit Valuation Adjustment at t, seen from the viewpoint of party k.

The word “Total” should reflect the total coverage of all four possible cases: $\Omega = N \,\dot\cup\, A_0^- \,\dot\cup\, A_2^- \,\dot\cup\, A_{\text{sim}}$.

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself,

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$.

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$.
Thus, "$\text{TDVA}_t(k, 2-k)$" is **not a typo**!

TCVA, TDVA and TBVA III

Note that TDVA$_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$. Thus, "TDVA$_t(k, 2-k)$" is **not a typo**! A proper (and more "network-adequate") notation probably could be the following one:

$$\text{“}\,\mathrm{TDVA}_t(k, 2-k \big| k) := \mathrm{TCVA}_t(2-k, k \big| k)\,\text{”}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)?

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$. Thus, "$\text{TDVA}_t(k, 2-k)$" is **not a typo**! A proper (and more "network-adequate") notation probably could be the following one:

$$\text{“}\,\text{TDVA}_t(k, 2-k \big| k) := \text{TCVA}_t(2-k, k \big| k)\,\text{''}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)? So, aren't we actually confronted with the beginning of an "accounting paradox" here?

TCVA, TDVA and TBVA III

Note that TDVA$_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$. Thus, "TDVA$_t(k, 2-k)$" is **not a typo**! A proper (and more "network-adequate") notation probably could be the following one:

$$\text{“}\,\mathrm{TDVA}_t(k, 2-k \,|\, k) := \mathrm{TCVA}_t(2-k, k \,|\, k)\,\text{''}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)? So, aren't we actually confronted with the beginning of an "accounting paradox" here? Even more puzzling: don't we simply ignore the right sign here and just view on k's double ledger k's paid default protection premium TCVA$_t(2-k|k)$ as a received amount, simply by renaming TCVA$_t(2-k|k)$ as "TDVA$_t(k, 2-k)$"?

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$. Thus, "$\text{TDVA}_t(k, 2-k)$" is **not a typo**! A proper (and more "network-adequate") notation probably could be the following one:

$$\text{“}\,\text{TDVA}_t(k, 2-k \big| k) := \text{TCVA}_t(2-k, k \big| k)\,\text{''}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)? So, aren't we actually confronted with the beginning of an "accounting paradox" here? Even more puzzling: don't we simply ignore the right sign here and just view on k's double ledger k's paid default protection premium $\text{TCVA}_t(2-k|k)$ as a received amount, simply by renaming $\text{TCVA}_t(2-k|k)$ as "$\text{TDVA}_t(k, 2-k)$"? Shouldn't it be "$\text{TDVA}_t(k, 2-k) := -\text{TCVA}_t(2-k|k)$"?

TCVA, TDVA and TBVA III

Note that $\text{TDVA}_t(k, 2-k)$ is defined contingent of the default of party k itself, yet **not** contingent of the default of party $2-k$. Thus, "$\text{TDVA}_t(k, 2-k)$" is **not a typo**! A proper (and more "network-adequate") notation probably could be the following one:

$$\text{“}\,\text{TDVA}_t(k, 2-k \,|\, k) := \text{TCVA}_t(2-k, k \,|\, k)\,\text{''}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)? So, aren't we actually confronted with the beginning of an "accounting paradox" here? Even more puzzling: don't we simply ignore the right sign here and just view on k's double ledger k's paid default protection premium $\text{TCVA}_t(2-k|k)$ as a received amount, simply by renaming $\text{TCVA}_t(2-k|k)$ as "$\text{TDVA}_t(k, 2-k)$"? Shouldn't it be "$\text{TDVA}_t(k, 2-k) := -\text{TCVA}_t(2-k|k)$"? Confused!

TCVA, TDVA and TBVA IV

TCVA, TDVA and TBVA IV

Observation

Let $k = 0$ *or* $k = 2$, $t \in [0, T]$ *and* $t \leq \tau^*$.

TCVA, TDVA and TBVA IV

University of Southampton
School of Mathematics

Observation

Let $k=0$ *or* $k=2$, $t\in[0,T]$ *and* $t\leq\tau^*$.

(i) *If* $A_{sim}=\emptyset$ $\mathbb{Q}$*-a. s., then we have (*$\mathbb{Q}$*-a. s.)*

$$TCVA_t(k\,|\,2-k)=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-}LGD_kD(t,\tau^*)(M_{2-k}(\tau^*))^+\middle|\mathcal{G}_t\right]\geq 0$$

TCVA, TDVA and TBVA IV

Observation

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$.

(i) *If $A_{sim} = \emptyset$ $\mathbb{Q}$-a. s., then we have ($\mathbb{Q}$-a. s.)*

$$TCVA_t(k \,|\, 2-k) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} LGD_k D(t, \tau^*)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right] \geq 0$$

(ii) *If $A_k^- = \emptyset$ $\mathbb{Q}$-a. s., then we have ($\mathbb{Q}$-a. s.)*

$$TCVA_t(k \,|\, 2-k) = -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{sim}} R_k D(t, \tau^*)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right] \leq 0$$

TCVA, TDVA and TBVA IV

Observation

Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \le \tau^*$.

(i) *If $A_{sim} = \emptyset$ $\mathbb{Q}$-a. s., then we have ($\mathbb{Q}$-a. s.)*

$$TCVA_t(k \mid 2-k) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} LGD_k D(t, \tau^*)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right] \geq 0$$

(ii) *If $A_k^- = \emptyset$ $\mathbb{Q}$-a. s., then we have ($\mathbb{Q}$-a. s.)*

$$TCVA_t(k \mid 2-k) = -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{sim}} R_k D(t, \tau^*)(M_{2-k}(\tau^*))^+ \middle| \mathcal{G}_t\right] \leq 0$$

So, what would (ii) say if in addition both parties did default simultaneously? Does then (ii) just state that the required partial reimbursement $R_k(M_{2-k}(\tau^*))^+$ of party $2-k$ by party k could be ignored by party k then?

Market price of total bilateral CCR I

Recall that we also have assumed that the discount factor process $D(0,\cdot)$ is $\mathbf{G}$-adapted.

Market price of total bilateral CCR I

Recall that we also have assumed that the discount factor process $D(0,\cdot)$ is $\mathbf{G}$-adapted. Let's sum up what we have seen so far:

Market price of total bilateral CCR I

Recall that we also have assumed that the discount factor process $D(0,\cdot)$ is $\mathbf{G}$-adapted. Let's sum up what we have seen so far:

Theorem

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - \mathit{TBVA}_t(k),$$

Market price of total bilateral CCR I

Recall that we also have assumed that the discount factor process $D(0,\cdot)$ is $\mathbf{G}$-adapted. Let's sum up what we have seen so far:

Theorem

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \le \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - \mathit{TBVA}_t(k),$$

where the stopped process $D(0,\cdot \wedge \tau^*)\,\mathit{TBVA}^{\tau^*}(k, 2-k)$ is a $\mathbf{G}$-martingale under $\mathbb{Q}$ (seen from the viewpoint of party k), satisfying $\mathit{TBVA}^{\tau^*}(k, 2-k) = -\mathit{TBVA}^{\tau^*}(2-k, k)$.

Market price of total bilateral CCR II

Theorem (ctd.)

Moreover,

$$TBVA_{\tau^*}(k, 2-k) = B_{\tau^*}(k, 2-k),$$

where $B_{\tau^*}(k, 2-k)$ is a $\mathbf{G}$-adapted stochastic process, satisfying the ISDA CCR free restrictions and $B_\cdot(k, 2-k) = -B_\cdot(2-k, k)$.

Market price of total bilateral CCR II

Theorem (ctd.)

Moreover,

$$\mathit{TBVA}_{\tau^*}(k, 2-k) = B_{\tau^*}(k, 2-k),$$

where $B_{\tau^*}(k, 2-k)$ *is a* $\mathbf{G}$*-adapted stochastic process, satisfying the ISDA CCR free restrictions and* $B_\cdot(k, 2-k) = -B_\cdot(2-k, k)$*.*

In particular, we have $\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] = -\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]}|\mathcal{G}_t\right]$, implying that both parties 0 and 2 would then agree on the "$\mathbb{Q}$-market price of total bilateral CCR".

Market price of total bilateral CCR II

Theorem (ctd.)

Moreover,

$$TBVA_{\tau^*}(k, 2-k) = B_{\tau^*}(k, 2-k),$$

where $B_{\tau^*}(k, 2-k)$ is a $\mathbf{G}$-adapted stochastic process, satisfying the ISDA CCR free restrictions and $B_\cdot(k, 2-k) = -B_\cdot(2-k, k)$.

In particular, we have $\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T)}|\mathcal{G}_t\right] = -\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T)}|\mathcal{G}_t\right]$, implying that both parties 0 and 2 would then agree on the "$\mathbb{Q}$-market price of total bilateral CCR".

In fact, we not only have an existence result. The listed properties already lead to the following "uniqueness" result:

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$.

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$*. Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$*, where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k*.*

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

- $\Delta_{\tau^*}(0, 2) = -\Delta_{\tau^*}(2, 0)$ *for all* $s \in [0, T]$*;*

Market price of total bilateral CCR III

Theorem

Let $k \in \{0,2\}$ *and* $t \in [0,T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

- $\Delta_{\tau^*}(0,2) = -\Delta_{\tau^*}(2,0)$ *for all* $s \in [0,T]$;
- *Both,* $\Delta_{\cdot}(0,2)$ *and* $\Delta_{\cdot}(2,0)$ *satisfy the ISDA CCR free restrictions at* τ^*;

Market price of total bilateral CCR III

Theorem

Let $k \in \{0,2\}$ *and* $t \in [0,T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

- $\Delta_{\tau^*}(0,2) = -\Delta_{\tau^*}(2,0)$ *for all* $s \in [0,T]$;
- *Both,* $\Delta_\cdot(0,2)$ *and* $\Delta_\cdot(2,0)$ *satisfy the ISDA CCR free restrictions at* τ^*;
- $D(0,\cdot)$ *is* $\mathbf{G}$*-adapted,*

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

- $\Delta_{\tau^*}(0, 2) = -\Delta_{\tau^*}(2, 0)$ *for all* $s \in [0, T]$;
- *Both,* $\Delta_\cdot(0, 2)$ *and* $\Delta_\cdot(2, 0)$ *satisfy the ISDA CCR free restrictions at* τ^*;
- $D(0, \cdot)$ *is* $\mathbf{G}$*-adapted,*
- *The stopped process* $D(0, \cdot \wedge \tau^*)\Delta^{\tau^*}(k, 2-k)$ *is a càdlàg* $\mathbf{G}$*-martingale.*

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$. *Consider* $Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$, *where (as usual in this talk)* $M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party* k. *Assume that*

- $\Delta_{\tau^*}(0, 2) = -\Delta_{\tau^*}(2, 0)$ *for all* $s \in [0, T]$;
- *Both,* $\Delta_\cdot(0, 2)$ *and* $\Delta_\cdot(2, 0)$ *satisfy the ISDA CCR free restrictions at* τ^*;
- $D(0, \cdot)$ *is* $\mathbf{G}$*-adapted,*
- *The stopped process* $D(0, \cdot \wedge \tau^*)\Delta^{\tau^*}(k, 2-k)$ *is a càdlàg* $\mathbf{G}$*-martingale.*

If the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, *we have* $\Delta_t(k, 2-k) = TBVA_t(k, 2-k)$ *and* $Z_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right]$.

1. Simultaneous defaults and total bilateral counterparty credit risk
2. Total bilateral valuation adjustment
3. Unilateral CVA and Basel III

Basel III and unilateral CVA

The confusion continues since:

Basel III and unilateral CVA

The confusion continues since:

Special Case (**Basel III** $\rightsquigarrow$ only one party defaults!)

Fix $k \in \{0, 2\}$. *Assume that in addition* $\tau_k = +\infty$ (*i. e., no default of party* k). *Then* $A^{-}_{2-k} = \{\tau_{2-k} \leq T\}$, $A^{-}_{k} = \emptyset$ *and* $A_{sim} = \emptyset$. *Consequently,* $TDVA_t(k, 2-k) = 0$,

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - TCVA_t(k \,|\, 2-k)\,,$$

Basel III and unilateral CVA

The confusion continues since:

Special Case (**Basel III** $\rightsquigarrow$ only one party defaults!)

Fix $k \in \{0, 2\}$. *Assume that in addition* $\tau_k = +\infty$ (*i. e., no default of party* k). *Then* $A^-_{2-k} = \{\tau_{2-k} \le T\}$, $A^-_k = \emptyset$ *and* $A_{sim} = \emptyset$. *Consequently,* $TDVA_t(k, 2-k) = 0$,

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T)} | \mathcal{G}_t\right] = M_t(k) - TCVA_t(k \,|\, 2-k)\,,$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T)} | \mathcal{G}_t\right] = M_t(2-k) + TDVA_t(2-k, k).$$

Basel III and unilateral CVA

The confusion continues since:

Special Case (**Basel III** $\rightsquigarrow$ only one party defaults!)

Fix $k \in \{0, 2\}$. Assume that in addition $\tau_k = +\infty$ (i. e., no default of party k). Then $A^-_{2-k} = \{\tau_{2-k} \leq T\}$, $A^-_k = \emptyset$ and $A_{sim} = \emptyset$. Consequently, $TDVA_t(k, 2-k) = 0$,

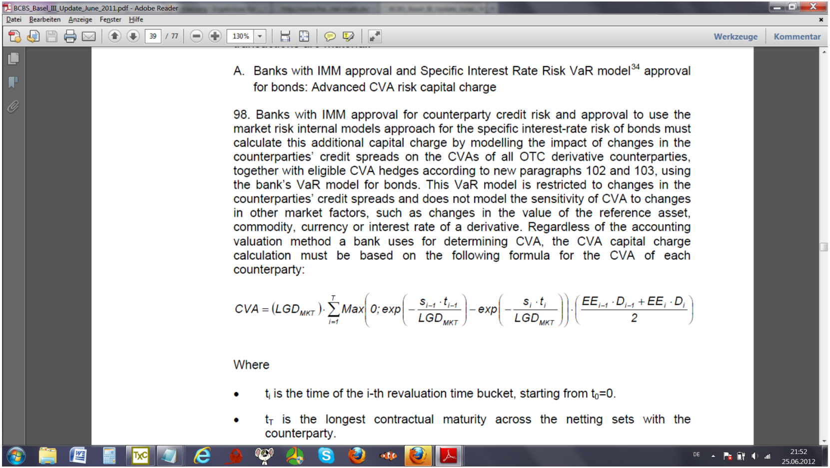
$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - TCVA_t(k \,|\, 2-k)\,,$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]} | \mathcal{G}_t\right] = M_t(2-k) + TDVA_t(2-k, k).$$

Hence, if party k were the investor, and if $\tau_k = +\infty$ the unilateral CVA $UCVA_t(k \,|\, 2-k) := TCVA_t(k \,|\, 2-k)$ would have to be paid by party $2-k$ to the default free party k at t to cover a potential default of party $2-k$ after t.

Excerpt from Basel III (ACVA, Para 98)



A. Banks with IMM approval and Specific Interest Rate Risk VaR model[34] approval for bonds: Advanced CVA risk capital charge

98. Banks with IMM approval for counterparty credit risk and approval to use the market risk internal models approach for the specific interest-rate risk of bonds must calculate this additional capital charge by modelling the impact of changes in the counterparties' credit spreads on the CVAs of all OTC derivative counterparties, together with eligible CVA hedges according to new paragraphs 102 and 103, using the bank's VaR model for bonds. This VaR model is restricted to changes in the counterparties' credit spreads and does not model the sensitivity of CVA to changes in other market factors, such as changes in the value of the reference asset, commodity, currency or interest rate of a derivative. Regardless of the accounting valuation method a bank uses for determining CVA, the CVA capital charge calculation must be based on the following formula for the CVA of each counterparty:

$$CVA = (LGD_{MKT}) \cdot \sum_{i=1}^{T} Max\left(0; exp\left(-\frac{s_{i-1} \cdot t_{i-1}}{LGD_{MKT}}\right) - exp\left(-\frac{s_i \cdot t_i}{LGD_{MKT}}\right)\right) \cdot \left(\frac{EE_{i-1} \cdot D_{i-1} + EE_i \cdot D_i}{2}\right)$$

Where

- t_i is the time of the i-th revaluation time bucket, starting from t_0=0.
- t_T is the longest contractual maturity across the netting sets with the counterparty.

CVA risk in Basel III: Flaws I

An analysis of “CVA volatility risk” and its capitalisation should particularly treat the following serious flaws:

(i) CVA risk (and hedges) extend far beyond the risk of credit spread changes. It includes all risk factors that drive the underlying counterparty exposures as well as dependent interactions between counterparty exposures and the credit spreads of the counterparties (and their underyings). By solely focusing on credit spreads, the Basel III UCVA VaR and stressed VaR measures in its advanced approach for determining a CVA risk charge do not reflect the real risks that drive the P&L and earnings of institutes. Moreover, banks typically hedge these non-credit-spread risk factors. The Basel III capital calculation does not include these hedges.

CVA risk in Basel III: Flaws II

(ii) The non-negligible and non-trivial problem of a more realistic inclusion of WWR should be analysed deeply. In particular, the “alpha” multiplier $1.2 \leq \alpha$ should be revisited, and any unrealistic independence assumption should be strongly avoided.

(iii) Credit and market risks in UCVA are not different from the same risks, embedded in many other trading positions such as corporate bonds, CDSs, or equity derivatives. CVA risk can be seen as just another source of market risk. Consequently, it should be managed within the trading book. Basel III requires that the CVA risk charge is calculated on a stand alone basis, separated from the trading book. This seems to be an artificial segregation. A suitable approach would be to include UCVA and all of its hedges into the trading book capital calculation.

CVA risk in Basel III: Flaws III

(iv) Basel III considers unilateral CVA only. More precisely, the regulatory calculation of the ACVA is based on UCVA_0 – as opposed to the calculations of CVA in FAS 157 respectively IAS 39! Latter explicitly include the (U)DVA_0. Hence, there exists a non-trivial mismatch between regulation and accounting! Moreover, as we have seen a thorough and appropriate treatment of a market price of (bilateral) CCR leads to TBVA_0 and not to UCVA_0. Consequently, further research is necessary. There is work in progress such as e.g. the running "Fundamental Review of the Trading Book" or running projects in the RTF subgroup of the BCBS – hopefully leading to necessary improvements of Basel III.

Structure of TCVA$_t(k \,|\, 2-k)$

Although we write "TCVA$_t(k \,|\, 2-k)$" it always should be kept in mind that we actually are working with a very complex object, namely:

$$\boxed{\text{TCVA}_k(t, T, \text{LGD}_{2-k}, \tau_k, \tau_{2-k}, D(t,\tau^*), M_k(\tau^*))} \quad !$$

Structure of $\text{TCVA}_t(k \mid 2-k)$

Although we write "$\text{TCVA}_t(k \mid 2-k)$" it always should be kept in mind that we actually are working with a very complex object, namely:

$$\boxed{\text{TCVA}_k(t, T, \text{LGD}_{2-k}, \tau_k, \tau_{2-k}, D(t,\tau^*), M_k(\tau^*))} \quad !$$

Basel III considers the case $t = 0$ "only". Why?

Structure of $\text{TCVA}_t(k \,|\, 2-k)$

University of Southampton School of Mathematics

Although we write "$\text{TCVA}_t(k \,|\, 2-k)$" it always should be kept in mind that we actually are working with a very complex object, namely:

$$\boxed{\text{TCVA}_k(t, T, \text{LGD}_{2-k}, \tau_k, \tau_{2-k}, D(t, \tau^*), M_k(\tau^*))} \quad !$$

Basel III considers the case $t = 0$ "only". Why? The case $0 < t \leq \tau^*$ requires an in depth analysis of the conditional joint default process

$$\Big(\mathbb{Q}\big(\tau_k \leq T \text{ and } \tau_{2-k} \leq \tau_k \big| \mathcal{G}_t\big)\Big)_{t \in [0,T]}.$$

Structure of $\text{TCVA}_t(k \mid 2-k)$

Although we write “$\text{TCVA}_t(k \mid 2-k)$” it always should be kept in mind that we actually are working with a very complex object, namely:

$$\boxed{\text{TCVA}_k(t, T, \text{LGD}_{2-k}, \tau_k, \tau_{2-k}, D(t, \tau^*), M_k(\tau^*))} \quad !$$

Basel III considers the case $t = 0$ “only”. Why? The case $0 < t \leq \tau^*$ requires an in depth analysis of the conditional joint default process

$$\Big(\mathbb{Q}\big(\tau_k \leq T \text{ and } \tau_{2-k} \leq \tau_k \big| \mathcal{G}_t\big)\Big)_{t \in [0,T]}.$$

To cover dynamically changing stochastic dependence between all embedded risk factors, a truly dynamic copula model has to be constructed ($\rightsquigarrow$ Bielecki, Crépey, Frey, Jeanblanc and many more).

Further important topics (not discussed here)

Finally let us list further – important – topics which could not be considered in this talk (due to time limitation).

Further important topics (not discussed here)

Finally let us list further – important – topics which could not be considered in this talk (due to time limitation).

- Risk mitigants such as collateral and margins;

Further important topics (not discussed here)

Finally let us list further – important – topics which could not be considered in this talk (due to time limitation).

- Risk mitigants such as collateral and margins;
- Re-hypothecation of collateral and funding;

Further important topics (not discussed here)

Finally let us list further – important – topics which could not be considered in this talk (due to time limitation).

- Risk mitigants such as collateral and margins;
- Re-hypothecation of collateral and funding;
- Margin period of risk;

Further important topics (not discussed here)

Finally let us list further – important – topics which could not be considered in this talk (due to time limitation).

- Risk mitigants such as collateral and margins;
- Re-hypothecation of collateral and funding;
- Margin period of risk;
- TBVA and clearing through a CCP (a CCP could also default) $\rightsquigarrow$ systemic risk?!

A very few references I

[1] C. Albanese, D. Brigo and F. Oertel.
Restructuring counterparty credit risk.
Discussion Paper, Deutsche Bundesbank, No 14/2013.

[2] S. Assefa, T. Bielecki, S. Crépey and M. Jeanblanc.
CVA computation for counterparty risk assessment in credit portfolios.
Credit Risk Frontiers. Editors T. Bielecki, D. Brigo and F. Patras, John Wiley & Sons (2011).

[3] T. F. Bollier and E. H. Sorensen.
Pricing swap default risk.
Financial Analysts Journal Vol. **50**, No. 3, pp. 23-33 (1994).

[4] D. Brigo and A. Capponi.
Bilateral counterparty risk valuation with stochastic dynamical models and application to credit default swaps.
Working Paper, Fitch Solutions and CalTech (2009).

A very few references II

[5] J. Gregory.
Counterparty Credit Risk.
John Wiley & Sons Ltd (2010).

[6] J. Gregory.
Being two-faced over counterparty credit risk.
Risk, February 2009 (2009).

[7] J. Hull and A. White.
CVA and Wrong Way Risk.
Working Paper, Joseph L. Rotman School of Management, University of Toronto (2011).

[8] M. Pykhtin.
A Guide to Modelling Counterparty Credit Risk.
GARP Risk Review, 37, 16-22 (2007).

A very few references III

[9] H. Schmidt.
Basel III und CVA aus regulatorischer Sicht.
Kontrahentenrisiko. S. Ludwig, M. R. W. Martin und C. S. Wehn (Hrsg.), Schäfer-Pöschel (2012).

Thank you for your attention!

UNIVERSITY OF Southampton
School of Mathematics

Thank you for your attention!

Are there any questions, comments or remarks?