A few remarks on random jump measures

There are several ways to describe the (random) jump measure j_X of a general càdlàg stochastic process $X = (X_t)_{t\geq 0}$ (such as e. g. a Lévy process) and to describe stochastic integration with respect to j_X. To this end, we start with a crucial result from [3] (cf. also [4]), stating in particular that the set

$$\{\Delta X \neq 0\} := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \Delta X_t(\omega) \neq 0\}$$

is a *thin* set.

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is a *thin* set.

More precisely, given the standing assumption that we fix an arbitrary filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ which satisfies the usual conditions, we have:

Theorem (Dellacherie, 1972)

Let $X = (X_t)_{t\geq 0}$ *be an arbitrary* $\mathbf{F}$*-adapted càdlàg stochastic process on* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$*. Then there exist a sequence* $(T_n)_{n\in\mathbb{N}}$ *of* $\mathbf{F}$*-stopping times such that* $[\![T_n]\!] \cap [\![T_k]\!] = \emptyset$ *for all* $n \neq k$ *and*[1]

$$\{\Delta X \neq 0\} = \dot{\bigcup_{n=1}^{\infty}} [\![T_n]\!] . \qquad (*)$$

In particular, $\Delta X_{T_n(\omega)}(\omega) \neq 0$ *for all* $\omega \in \Omega$ *and* $n \in \mathbb{N}$*.*

Definition

Let X be as above. Any sequence of $\mathbf{F}$-stopping times $(T_n)_{n\in\mathbb{N}}$ satisfying the representation $(*)$ is called an exhausting representation of the set $\{\Delta X \neq 0\}$.

[1] If T is a stopping time, $[\![T]\!] := \{(s,\omega) : T(\omega) = s\}$ denotes the (reflected) graph of T.

In the following let us assume that $X_{0-} := X_0 := 0$.

Corollary

Let $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ *and* $\omega \in \Omega$. *Then*

$$j_X(\omega, B) \quad = \quad \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_B(s,x) j_X(\omega, d(s,x))$$

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$$\begin{aligned} j_X(\omega, B) &= \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_B(s,x) j_X(\omega, d(s,x)) \\ &= \#\{s > 0 : \Delta X_s(\omega) \neq 0 \text{ and } \big(s, \Delta X_s(\omega)\big) \in B\} \end{aligned}$$

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&= \sum_{s>0} \mathbb{1}_B\big(s, \Delta X_s(\omega)\big) \mathbb{1}_{\mathbb{R}^*}(\Delta X_s(\omega)) \\
&= \sum_{n=1}^{\infty} \mathbb{1}_B\big(T_n(\omega), \Delta X_{T_n(\omega)}(\omega)\big)
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Proof.

Since $1\!\!1_{\{\Delta X \neq 0\}}(s,\omega) = \sum_{n=1}^{\infty} 1\!\!1_{[\![T_n]\!]}(s,\omega) = \sum_{n=1}^{\infty} 1\!\!1_{\{T_n(\omega)\}}(s)$, we just have to permute the two sums. □

A constructive (and hence implementable) exhausting representation of a thin set of “entered jumps” is given in the following important special case:

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Proposition

Let $X = (X_t)_{t \geq 0}$ be an arbitrary $\mathbf{F}$-adapted càdlàg stochastic process on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ and $A \in \mathcal{B}(\mathbb{R})$ such that $0 \notin \overline{A}$. Put

$$T_1^A(\omega) := \inf\{t > 0 : \Delta X(\omega) \in A\}$$

and

$$T_n^A(\omega) := \inf\{t > T_{n-1}^A(\omega) : \Delta X(\omega) \in A\} \qquad (n \geq 2).$$

Up to an evanescent set $(T_n^A)_{n \in \mathbb{N}}$ defines a sequence of strictly increasing $\mathbf{F}$-stopping times, satisfying

$$\{\Delta X \in A\} = \dot{\bigcup_{n=1}^{\infty}} [\![S_n^A]\!] \ , \qquad (**)$$

where

Proposition (ctd.)

$$S_n^A := T_n^A \, \mathbb{1}_A\left(\Delta X_{T_n^A}\right) + (+\infty)\, \mathbb{1}_{A^c}\left(\Delta X_{T_n^A}\right). \qquad (n \in \mathbb{N})$$

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In virtue of [6] (Chapter 4, p. 25ff) each T_n^A is a $\mathbf{F}$-stopping time and $\Omega_0 \times \mathbb{R}_+$ is an evanescent set, where $\Omega_0 := \{\omega \in \Omega : \lim_{n\to\infty} T_n^A(\omega) < \infty\}$. Fix $(\omega, t) \notin \Omega_0 \times \mathbb{R}_+$.

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Assume by contradiction that $T_{m_0}^A(\omega) = T_{m_0+1}^A(\omega) =: t^*$ for some $m_0 \in \mathbb{N}$. By definition of $t^* = T_{m_0+1}^A(\omega)$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ $\lim_{n\to\infty} t_n = t^*$, $\Delta X_{t_n}(\omega) \in A$, and $t^* = T_{m_0}^A(\omega) < t_{n+1} \leq t_n$. Consequently, since X has right-continuous paths, it follows that $\Delta X_{t^*}(\omega) = \lim_{n\to\infty} \Delta X_{t_n}(\omega) \in \bar{A}$, implying that $\Delta X_{t^*}(\omega) \neq 0$ (since $0 \notin \overline{A}$). Thus $\lim_{n\to\infty} t_n = t^*$ is an accumulation point of the at most countable set $\{t > 0 : \Delta X_t(\omega) \neq 0\}$ - a contradiction.

Proof (ctd).

To prove the set equality $(**)$, let firstly $\Delta X_t(\omega) \in A$. Assume by contradiction that for all $m \in \mathbb{N}$ $T_m^A(\omega) \neq t$. Since $\omega \notin \Omega_0$, there is some $m_0 \in \mathbb{N} \cap [2, \infty)$ such that $T_{m_0}^A(\omega) > t$. Choose m_0 small enough, so that $T_{m_0-1}^A(\omega) \leq t < T_{m_0}^A(\omega)$. Consequently, since $\Delta X_t(\omega) \in A$, we must have $t \leq T_{m_0-1}^A(\omega)$ and hence $T_{m_0-1}^A(\omega) = t$. However, the latter contradicts our assumption. Thus, $\{\Delta X \in A\} \subseteq \bigcup_{n=1}^{\infty} [\![T_n^A]\!]$.

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The claim now follows from [5], Theorem 3.19. □

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The claim now follows from [5], Theorem 3.19. □

Note that $\{S_n^A < +\infty\} \subseteq \{\Delta X_{T_n^A} \in A\} \subseteq \{S_n^A = T_n^A\}$. Hence,

$$\mathbb{1}_A\left(\Delta X_{T_n^A}\right)\mathbb{1}_{\{T_n^A \leq t\}} = \mathbb{1}_{\{S_n^A \leq t\}}$$

for all $n \in \mathbb{N}$.

Keeping the setup of the last Proposition in mind, we now are going to consider an important special case of a Borel set B on $\mathbb{R}_+ \times \mathbb{R}$, leading to the construction of “stochastic” integrals with respect to the jump measure j_X including the construction of stochastic jump processes which play a fundamental role in the theory and application of Lévy processes.

Keeping the setup of the last Proposition in mind, we now are going to consider an important special case of a Borel set B on $\mathbb{R}_+ \times \mathbb{R}$, leading to the construction of "stochastic" integrals with respect to the jump measure j_X including the construction of stochastic jump processes which play a fundamental role in the theory and application of Lévy processes.

To this end, let us consider all Borel sets B on $\mathbb{R}_+ \times \mathbb{R}$ of type $B = [0, t] \times A$, where $t \geq 0$ and

$$A \in \mathcal{B}^* := \{A : A \in \mathcal{B}(\mathbb{R}), 0 \notin \overline{A}\} .$$

Obviously, $A \subseteq \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ for *all* $\varepsilon > 0$, implying in particular that $A \in \mathcal{B}^*$ is bounded from below.

Lemma (cf. [1], Lemma 2.3.4.)

Let $X = (X_t)_{t \geq 0}$ *be a càdlàg process. Let* $A \in \mathcal{B}^*$ *and* $t > 0$*. Then* $N_X^A(t) := j_X(\cdot, [0, t] \times A) < \infty$ *a. s.*

Proposition

Let $X = (X_t)_{t\geq 0}$ *be a càdlàg process and* $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *be measurable. Let* $A \in \mathcal{B}^*$ *and* $t > 0$*. Then for all* $\omega \in \Omega$ *the function* $\mathbb{1}_{[0,t]\times A} f$ *is a. s. integrable with respect to the jump measure* $j_X(\omega, d(s,x))$*, and*

$$\int_{[0,t]\times A} f(s,x) j_X(\omega, d(s,x))$$

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$$\int_{[0,t] \times A} f(s, x) j_X(\omega, d(s, x)) = \sum_{0 < s \leq t} f\big(s, \Delta X_s(\omega)\big) \mathbb{1}_A(\Delta X_s(\omega))$$

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Let $X = (X_t)_{t\geq 0}$ be a càdlàg process and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be measurable. Let $A \in \mathcal{B}^*$ and $t > 0$. Then for all $\omega \in \Omega$ the function $\mathbb{1}_{[0,t]\times A} f$ is a. s. integrable with respect to the jump measure $j_X(\omega, d(s,x))$, and

$$\int_{[0,t]\times A} f(s,x) j_X(\omega, d(s,x))$$

$$= \sum_{0<s\leq t} f\big(s, \Delta X_s(\omega)\big) \mathbb{1}_A(\Delta X_s(\omega))$$

$$= \sum_{n=1}^{\infty} f\big(T_n(\omega), \Delta X_{T_n(\omega)}(\omega)\big) \mathbb{1}_A\big(\Delta X_{T_n(\omega)}\big) \mathbb{1}_{\{T_n\leq t\}}(\omega).$$

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$$\begin{aligned} & \int_{[0,t]\times A} f(s,x) j_X(\omega, d(s,x)) \\ = \quad & \sum_{0<s\leq t} f\big(s, \Delta X_s(\omega)\big)\mathbb{1}_A(\Delta X_s(\omega)) \\ = \quad & \sum_{n=1}^{\infty} f\big(T_n(\omega), \Delta X_{T_n(\omega)}(\omega)\big)\mathbb{1}_A\big(\Delta X_{T_n(\omega)}\big)\mathbb{1}_{\{T_n\leq t\}}(\omega). \end{aligned}$$

Moreover, given $\omega \in \Omega$ *there exists* $c_t^A(\omega) \in [0,\infty)$ *such that*

$$\int_{[0,t]\times A} |f(s,x)| j_X(\omega, d(s,x)) \leq c_t^A(\omega)\, j_X(\omega, [0,t]\times A)\,.$$

Proof.

Fix $\omega \in \Omega$ and consider the measurable function $g_t^A := 1\!\!1_{[0,t]\times A} f$.
Then $\mathbb{R}_+ \times \mathbb{R} = B_1(\omega) \dot\cup B_2(\omega)$, where
$B_1(\omega) := \{(s, \Delta X_s(\omega) : s > 0\}$ and $B_2(\omega) := \mathbb{R}_+ \times \mathbb{R} \setminus B_1(\omega)$.
Obviously, we have

$$j_X(\omega, B_2(\omega)) = \sum_{s>0} 1\!\!1_{B_2(\omega)}\big(s, \Delta X_s(\omega)\big) 1\!\!1_{\mathbb{R}^*}(\Delta X_s(\omega)) = 0\,,$$

implying that $I_2 := \int_{B_2(\omega)} |g_t^A(s,x)| j_X(\omega, d(s,x)) = 0$. Put
$I_1 := \int_{B_1(\omega)} |g_t^A(s,x)| j_X(\omega, d(s,x))$. Since on $[0,t]$ the càdlàg path $s \mapsto X_s(\omega)$ has only finitely many jumps in $A \in \mathcal{B}^*$ there exist finitely many elements $(s_1, \Delta X_{s_1}(\omega)), \ldots, (s_N, \Delta X_{s_N}(\omega))$ which all are elements of $\big([0,t] \times A\big) \cap B_1(\omega)$ (for some $N = N(\omega, t, A) \in \mathbb{N}$). Put

$$0 \le c_t^A(\omega) := \max_{1 \le k \le N} |f(s_k, \Delta X_{s_k}(\omega))| < \infty\,.$$

Proof (ctd).

Then

$$|g_t^A| = 1\!\!1_{[0,t]\times A}\,|f| \le c_t^A(\omega)\,1\!\!1_{[0,t]\times A} \text{ on } B_1\,,$$

and it follows that $I_2 \le c_t^A(\omega)\,j_X(\omega,[0,t]\times A)$. A standard monotone class argument finishes the proof. □

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Remark

Note that in terms of the previously discussed stopping times S_n^A we may write

$$\int_{[0,t]\times A} f(s,x) j_X(\omega, d(s,x)) = \sum_{n=1}^{\infty} f\left(S_n^A(\omega), \Delta X_{S_n^A(\omega)}(\omega)\right)\mathbb{1}_{\{S_n^A \leq t\}}(\omega)\,.$$

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In the case of a Lévy process X the following important special cases $f(s,x) := 1$ and $f(s,x) := x$ are embedded in the following non-trivial result:

Theorem

Let $X = (X_t)_{t\geq 0}$ be a (càdlàg) Lévy process and $A \in \mathcal{B}^*$.

(i) *Given $t \geq 0$*

$$\begin{aligned} N_X^A(t) &= \int_A N_X^{dx}(t) := j_X(\cdot, [0,t] \times A) = \int_{[0,t]\times A} j_X(\cdot, d(s,x)) \\ &= \sum_{0<s\leq t} \mathbb{1}_A(\Delta X_s) = \sum_{n=1}^{\infty} \mathbb{1}_A\left(\Delta X_{T_n}\right)\mathbb{1}_{\{T_n \leq t\}} = \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n^A \leq t\}} \end{aligned}$$

induces a Poisson process $N_X^A = \left(N_X^A(t)\right)_{t\geq 0}$ with intensity measure $\nu_X(A) := \mathbb{E}[N_X^A(1)] < \infty$.

Theorem (ctd.)

(ii) *Given* $t \geq 0$ *and a Borel measurable function* $g : \mathbb{R} \longrightarrow \mathbb{R}$

$$
\begin{aligned}
Z_X^A(t) &:= \int_A g(x)\, N_X^{dx}(t) = \int_{[0,t]\times A} g(x)\, j_X(\cdot, d(s,x)) \\
&= \sum_{0<s\leq t} g\big(\Delta X_s\big)\, \mathbb{1}_A(\Delta X_s) = \sum_{n=1}^{\infty} g\big(\Delta X_{T_n}\big) \mathbb{1}_A\big(\Delta X_{T_n}\big) \mathbb{1}_{\{T_n \leq t\}} \\
&= \sum_{n=1}^{\infty} g\big(\Delta X_{S_n^A}\big) \mathbb{1}_{\{S_n^A \leq t\}} = \sum_{n=1}^{N_X^A(t)} g\big(\Delta X_{S_n^A}\big)
\end{aligned}
$$

induces a compound Poisson process $Z_X^A = \big(Z_X^A(t)\big)_{t\geq 0}$.
Moreover, if $g \in L^1(A, \nu_X)$ *then* $\mathbb{E}[Z_X^A(t)] = t\nu_X(A)\mathbb{E}[g\big(\Delta X_{S_1^A}\big)]$.

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