A first approach to randomness in financial markets with a view towards option pricing

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A Daily Problem in Risk Management

Today, at time t = 0, an American company XYZ agrees to deliver machine parts to Ireland in 9 months (at time T).

Unknown \$-price of Euros at time T constitutes a risk for this company because the price can fall!

Today, in order to reduce that risk, the company XYZ buys a right to sell Euros at time T at a \$-price which is fixed today, at time $t = 0$ (called the *strike price*). This is a typical example of a derivative security or a contingent claim (CC), called an European Put Option (EPO).

Problem: What is the price of an EPO?

A Single–Period Option Pricing Model

- Two trading dates: "Now" ($t = 0$) and "later" ($t = 1$);

- Two assets: A risk–free asset (zero–coupon bond) and a risky asset (here, it is the price of 1 Euro in Dollars).

More precisely:

- Let $r \geq 0$ be a risk–free interest rate which is assumed to be constant during the whole time period $[0, 1]$ and $B = (B_0, B_1)$ be the \$–price process of a related zero–coupon bond so that $B_0 = 1$ and $B_1 = 1 + r$;

- Let $X = (X_0, X_1)$ denote the price process of a risky asset (here, at time t, 1 Euro $= X_t$ \$);

- Two possible states for the random variable X_1 only: The price either moves up to $X_1(\omega_1)$ with subjective probability p or down to $X_1(\omega_2)$ with subjective probability $1 - p$.

Remark 1 *We need not restrict our model to the case (B, X) only. Indeed, adding N risky assets $S^1, \ldots, S^N$ to the riskless security $S^0 := B$ (where $N \in \mathbb{N}$ can be arbitrarily large), the SPM (B, X) is a special case of the general SPM $(S^0, S^1, \ldots, S^N)$.*

The data set: $X_0 = 1.150$, $X_1(\omega_1) = 1.456$, $X_1(\omega_2) = 0.844$, $r = 2\% = 0.02$ and $p := 3/8 = 0.375$

The payoff profile of the EPO at $t = 1$:

$$
\begin{aligned}
H(\omega) &= (X_0 - X_1(\omega))^+ \\
&= \max\{X_0 - X_1(\omega), 0\},
\end{aligned}
$$

which implies that

$$H(\omega_1) = 0 \text{ and } H(\omega_2) = 0.306.$$

Let $\pi(H)$ denote the price of the EPO at $t = 0$.

Intuitive Conjecture: *Classical probability tells us that the price $\pi(H)$ is given by the mean of the discounted value $\frac{H}{1+r}$ – calculated under the subjective probability $\mathbb{P} = (p, 1 - p)$:*

$$\begin{aligned}\pi(H) &\stackrel{?}{=} \mathbb{E}_{\boxed{\mathbb{P}}}\left(\frac{H}{1+r}\right) \\ &= p \cdot \frac{H(\omega_1)}{1+r} + (1-p) \cdot \frac{H(\omega_2)}{1+r} \\ &= 0.1875 \ldots ?!\end{aligned}$$

If a bank is selling 10000 EPOs at this price, it will make a good deal: It gains 485 \$ – without taking any risk!

Why?

Consider the following three balance sheets of the bank for a single EPO:

At $t = 0$	gain (\$)
Sell the EPO	$+\pi(H)$
Put 0.714 \$ on a savings account at rate $r = 2\%$	-0.714
Sell 0.500 Euros at price $X_0 = 1.15$ \$	$0.500 \cdot 1.15 = 0.575$
Balance at $t = 0$:	$\pi(H) - 0.139$

Since the Euro price X_0 either moves up to $X_1(\omega_1)$ or down to $X_1(\omega_2)$, two balance sheets must be considered at $t = 1$! Namely:

At $t = 1$ (given scenario ω_1)	gain (\$)
EPO is worthless	$\pm 0 = -H(\omega_1)$
Get back the loan of 0.714 \$ at rate $r = 2\%$	$0.714 \cdot 1.02 =$ $+0.728$
Buy 0.500 Euros at price $X_1(\omega_1) = 1.456$ \$	-0.728
Balance at $t = 1$:	± 0

At $t = 1$ (given scenario ω_2)	gain ($)
EPO is exercised	$-0.306 = -H(\omega_2)$
Get back the loan of 0.714 $ at rate $r = 2\%$	$0.714 \cdot 1.02 =$ $+0.728$
Buy 0.500 Euros at price $X_1(\omega_2) = 0.844$ $	-0.422
Balance at $t = 1$:	± 0

Thus, if for a single EPO at $t = 0$ the buyer assumes a price $\pi(H) = 0.1875 > 0.139$ $, the bank (the writer) can build up at $t = 0$ a portfolio which leads to a guaranteed profit at $t = 1$ – independent of the EPO's inherent risk:

- Buy 7140 riskless $–zerobonds at price $B_0 = 1$ $;

- Sell 5000 units of the underlying (i.e., Euros) at price $X_0 = 1.15$ $;

- Sell 10000 EPOs on the underlying X at price $10000\,\pi(H)$ $.

Portfolio Balance at $t = 0$:

$$10000\,\pi(H) + 5000\,X_0 - 7140\,B_0 = 485$$

Portfolio Balance at $t = 1$ (given an arbitrary scenario ω):

$$10000\,H(\omega) + 5000\,X_1(\omega) - 7140\,B_1(\omega) = 0$$

Objection: What happens if we even assume that $p = \frac{2}{3}$? In this case $\pi(H) = \mathbb{E}_{\mathbb{P}}(\frac{H}{1+r}) = 0.1 < 0.139\ldots$?

Exercise: Then the buyer of the option makes a good deal! Independent of the inherent risk of the EPO, she gains 485 $.

The only conclusion: To prevent such a situation which allows the writer or the buyer of the contingent claim to gain a profit without taking any risk, the price of the EPO necessarily is $\pi(H) = 0.139$ $ – independent of the subjective probability $\mathbb{P}$!

Put $a := \pi(H)$ and assume that $a = 0.1875$. Then we even obtain a money machine! Without paying anything at $t = 0$, we can construct a suitable portfolio which consists of a zero–coupon bond (with price B_0), the Euro (with price X_0) and the EPO (with price a) but which pays a strictly positive amount at $t = 1$, independent of the choosen scenario ω! How does this work?

Let $\varphi := (\varphi_0, \varphi_1, \varphi_2) \in \mathbb{R}^3$ arbitrary. Put $Y_0 := a$ and $Y_1 := H$. Consider the value process of the portfolio (B, X, Y):

$$V_t(\varphi) := \varphi_0 \, B_t + \varphi_1 \, X_t + \varphi_2 \, Y_t \quad (t = 0, 1).$$

Put $\psi_1 := 0.500$. Then, at $t = 0$, we construct the following profit strategy:

- $\varphi_0 := \psi_1 \, X_0 + Y_0 = 0.575 + a = 0.7625$ (sell 0.7625 zero–coupon bonds at 1 \$);

- $\varphi_1 := -\psi_1 = -0.500$ (buy 0.500 Euros at 1.15 \$);

- $\varphi_2 := -1$ (buy 1 EPO at price a).

Then:

$$\begin{aligned} V_0(\varphi) &= \varphi_0 B_0 + \varphi_1 X_0 + \varphi_2 Y_0 \\ &= 0.7625 - 0.575 - 0.1875 \\ &= 0 \end{aligned}$$

Without changing these portfolio–weights φ_0, φ_1 and φ_2 during the trading period $[0, 1]$, the portfolio value jumps to

$$\begin{aligned} V_1(\varphi)(\omega) &= \varphi_0 B_1 + \varphi_1 X_1(\omega) + \varphi_2 Y_1(\omega) \\ &= \varphi_0 (1 + r) + \varphi_1 X_1(\omega) + \varphi_2 H(\omega) \\ &= 0.778 + 0.500 X_1(\omega) - H(\omega) \\ &> 0 \quad \text{for all } \omega \in \{\omega_1, \omega_2\} \end{aligned}$$

Exercise: Any value $a \neq 0.139$ implies the existence of such a money machine!

Definition 1 *A trading strategy* $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^N) \in \mathbb{R}^{N+1}$ *in a general SPM which consists of a riskless zero–coupon bond* S^0 *and* N *risky assets* $S^1, \dots, S^N$ *is called an arbitrage strategy if the corresponding value process*

$$V_t(\varphi) := \sum_{n=0}^{N} \varphi^n S_t^n \quad (t = 0, 1)$$

satisfies the following properties:

- $V_0(\varphi) = 0$;
- $V_1(\varphi)(\omega) \geq 0 \ \ \forall \omega$;
- $\mathbb{E}(V_1(\varphi)) > 0$.

An SPM satisfies the No–Arbitrage–Property (NAP) if it does not contain any arbitrage strategies.

Problem: How can we calculate the portfolio–weights $\psi_0 :=$ 0.714 and $\psi_1 :=$ 0.500? Why should be $a = \pi(H) \approx$ 0.139? Do there exist hidden rules?

Theorem 1 (Law of One Price) *Assume that the extended market (B, X, Y) satisfies the NAP. Let $\varphi := (\varphi_0, \varphi_1, \varphi_2) \in \mathbb{R}^3$ so that*

$$V_1(\varphi)(\omega) := \varphi_0\, B_1 + \varphi_1\, X_1(\omega) + \varphi_2\, Y_1(\omega) = 0$$

for every scenario ω at $t = 1$. Then, at $t = 0$:

$$V_0(\varphi) := \varphi_0\, B_0 + \varphi_1\, X_0 + \varphi_2\, Y_0 = 0.$$

Proof: Assume that the statement is false. Then there exists a strategy $\psi := (\psi_0, \psi_1, \psi_2)$ so that $V_0(\psi) = \psi_0 + \psi_1\, X_0 + \psi_2\, Y_0 < 0$ and $V_1(\psi)(\omega) = \psi_0\,(1+r) + \psi_1\, X_1(\omega) + \psi_2\, Y_1(\omega) = 0$ for all ω. Put $\tilde{\varphi} := (-\psi_1\, X_0 - \psi_2\, Y_0,\ \psi_1,\ \psi_2)$.

Clearly, $V_0(\tilde{\varphi}) = 0$. Since $\psi_0 + \psi_1 X_0 + \psi_2 Y_0 < 0$, it follows that $\tilde{\varphi}_0 = -\psi_1 X_0 - \psi_2 Y_0 > \psi_0$ and therefore $V_1(\tilde{\varphi})(\omega) = \tilde{\varphi}_0 (1+r) + \tilde{\varphi}_1 X_1(\omega) + \tilde{\varphi}_2 Y_1(\omega) > \psi_0 (1+r) + \psi_1 X_1(\omega) + \psi_2 Y_1(\omega) = V_1(\psi)(\omega) = 0$ for all ω. Hence, $V_0(\tilde{\varphi}) = 0$ and $V_1(\tilde{\varphi})(\omega) > 0$ for all ω, which in particular implies that $\tilde{\varphi}$ is an arbitrage strategy – a contradiction. □

Our example: Let $\psi_0 :=$ 0.714 and $\psi_1 :=$ 0.500. Consider the extended market (B, X, Y) where $Y_0 := \pi(H) = a$ and $Y_1 := H$ and put $\varphi := (\psi_0, -\psi_1, -1)$. Then it follows that $V_1(\varphi) = 0$. Thus, if (B, X, Y) satisfies the NAP, $V_0(\varphi) \overset{!}{=} 0$ (due to Theorem 1). But:

$$V_0(\varphi) = 0 \iff a = \psi_0 B_0 - \psi_1 X_0 = \text{0.139}.$$

and

$$V_1(\varphi) = 0 \iff H = \psi_0 B_1 - \psi_1 X_1.$$

Conclusion: If a portfolio $\varphi := (\varphi_0, \varphi_1)$ replicates the EPO H in the market (B, X) and if the extended market (B, X, Y) has the NAP, then necessarily

$$\pi(H) = \varphi_0 \, B_0 + \varphi_1 \, X_0 = V_0(\varphi)$$

is the ("fair") price of H.

Definition 2 *A contingent claim H in a general SPM is called replicable if there exists a portfolio strategy $\varphi \in \mathbb{R}^{N+1}$ so that*

$$H(\omega) = V_1(\varphi)(\omega) \text{ for } \underline{\text{every}} \text{ scenario } \omega.$$

A general SPM is called complete, if every contingent claim is replicable. An incomplete SPM is a SPM which is not complete.

Theorem 2 *The SPM (B, X) is complete.*

Proof: Let H be an arbitrary contingent claim. The three vectors $\binom{1+r}{1+r}$, $\binom{u\,X_0}{d\,X_0}$ and $\binom{H(\omega_1)}{H(\omega_2)}$ are linearily dependent in $\mathbb{R}^2$ which implies the existence of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ so that

$$\binom{H(\omega_1)}{H(\omega_2)} = \alpha \binom{1+r}{1+r} + \beta \binom{u\,X_0}{d\,X_0}.$$

Solving this linear equation system gives the following solution:

$$\alpha = \frac{u\,H(\omega_2) - d\,H(\omega_1)}{(u-d)\,(1+r)}$$

and

$$\beta = \frac{H(\omega_1) - H(\omega_2)}{(u-d)\,X_0}$$

$\square$

Our example: Here, $u = \frac{1.456}{1.15} \approx 1.266$, $d = \frac{0.844}{1.15} \approx 0.734$ and $H = (X_0 - X_1)^+$. Thus: $\alpha =$ 0.714 $= \psi_0$ and $\beta =$ - 0.500 $= -\psi_1$!

Risk Neutral Evaluation

We consider our simple SPM (B, X). Let $0 < d < u$. An easy exercise shows the equivalence of the following statements:

- (B, X) has the NAP;
- $d < 1 + r < u$;
- $0 < q := \dfrac{(1+r) - d}{u - d} < 1$.

Thus, if we put $Q(\{\omega_1\}) := q$ and $Q(\{\omega_2\}) := 1 - q$ and if the SPM (B, X) satisfies the NAP, Q defines another probability measure! $Q \sim \mathbb{P}$ is called a *risk neutral probability measure* due to the following reason:

An investor who calculates w.r.t. Q (instead of the subjective probability $\mathbb{P}$) cannot differ between an investment in the riskless security B or an investment in the risky asset X! Namely, given any asset $S \in \{B, X\}$, consider its return $R_1(S) := \frac{S_1 - S_0}{S_0}$ at $t = 1$. Then $q = 0.538$ and

$$\mathbb{E}_Q(R_1(X)) = r = \mathbb{E}_Q(R_1(B))$$

$$\Updownarrow$$

$$\mathbb{E}_Q\Big(\frac{X_1}{B_1}\Big) = \frac{X_0}{B_0}.$$

Moreover:

$$\mathbb{E}_Q\Big(\frac{H}{B_1}\Big) \stackrel{!}{=} 0.139 = V_0(\psi_0, -\psi_1).$$

Summarizing our previous considerations leads to:

Theorem 3 *Let $0 < d < 1 + r < u$. Let $H = V_1(\varphi)$ be an arbitrary contingent claim in the SPM (B, X) with replicating portfolio φ. Put $Y_0 = \pi(H)$ and $Y_1 = H$. TFAE:*

- *The extendend market (B, X, Y) has the NAP;*

- $\pi(H) = \mathbb{E}_Q(\frac{H}{B_1})$;

- $\pi(H) = V_0(\varphi)$.

All the previous considerastions are special cases of the following non–trivial and very important Fundamental Theorems of Asset Pricing for SPMs which can be transferred to much more general financial market models:

Theorem 4 (1st FTAP) *Given an arbitrary general SPM* $(S^0, S^1, \dots, S^N)$*, the following statements are equivalent:*

(i) *The SPM* $(S^0, S^1, \dots, S^N)$ *satisfies the NAP;*

(ii) *There exists at least one risk neutral measure* Q*.*

Theorem 5 (2nd FTAP) *Given an arbitrary general SPM* $(S^0, S^1, \dots, S^N)$ *which satisfies the NAP, the following statements are equivalent:*

(i) *The SPM* $(S^0, S^1, \dots, S^N)$ *is complete;*

(ii) *There exists one and only one risk neutral measure* Q*.*

The Binomial Model of Cox–Ross–Rubinstein

- $T + 1$ trading dates $0, 1, \ldots, T$; Trading and portfolio selection during $[0, T]$ is possible at these dates only. During each time interval $(t-1, t)\,(t = 1, 2, \ldots T)$ the trading desk is closed.

- Two assets: a risk–free asset (zero–coupon bond) and a risky asset (a stock index, say).

More precisely:

- Let $r \geq 0$ be a risk–free interest rate which is assumed to be the same in each time interval $[t-1, t]$. Put $B_t := (1+r)^t$ so that $(B_0, B_1, \ldots, B_T)$ is the price process of a zero–coupon bond.

- Consider for each time interval $[t-1, t]$ the related SPM and link all the SPMs independently. Then we obtain a tree in the following sense: On a suitable probability space, the price process $(X_0, X_1, \ldots, X_T)$ of a risky asset is modeled as

$$X_t := X_0 \prod_{i=1}^{t} \xi_i, \qquad t = 0, 1, \ldots, T,$$

where X_0 is the known asset price (at $t = 0$) and $\xi_1, \ldots, \xi_T$ are independent and identically distributed rvs with subjective probability

$$\mathbb{P}(\xi_i = u) = p = 1 - \mathbb{P}(\xi_i = d).$$

Theorem 6 *Let* $0 < d < 1 + r < u$. *Then the CRR model satisfies the NAP and is complete. For any contingent claim* H *the risk neutral price* $\pi_t(H)$ *at time* $t \in \{0, 1, \ldots, T\}$ *is given by*

$$\pi_t(H) = B_t \cdot \mathbb{E}_Q(\frac{H}{B_T}|\mathcal{F}_t),$$

where $\mathbb{E}_Q$ *is the expectation operator with respect to the unique risk neutral probability measure (⇝ eqivalent martingale measure)* $q = \frac{(1+r)-d}{u-d}$ *and* $\mathcal{F}_t$ *represents the information about the asset price* X_t *at* t *(⇝ filtration).*

A particular case: An European Call Option (ECO) $H := (X_T - K)^+$. Put

$$t^* := \min\{t \in \mathbb{N}_0 : X_0\, u^t\, d^{T-t} > K\}.$$

Theorem 7 *Consider an ECO with expiry T and strike price K, written on (one share of) the stock X. Let $0 < d < 1 + r < u$. Then the risk neutral price of the option at $t = 0$ is given by*

$$\pi_0(H) = X_0 \cdot B(t^*, T; \tilde{p}) - \frac{K}{B^T} \cdot B(t^*, T; q),$$

where $\tilde{p} = \frac{q\,u}{1+r}$, $q = \frac{(1+r)-d}{u-d}$ and $B(\tau, T; p) := \sum_{t=\tau}^{T} \binom{T}{t} p^t (1-p)^{T-t} = \mathbb{P}(\tau \le Z \le T)$ with binomially distributed $Z \sim BIN(T; p)$ ($\tau \in \{0, 1, \ldots, T\}$ and $0 < p < 1$).

The Black–Scholes Model: From Trees to Brownian Motion

- Continous time interval $[0, T]$;

- Two assets: a risk–free asset (zero–coupon bond) and a risky asset (a stock index, say).

More precisely:

- Let $r \geq 0$ be a continuous short rate which is assumed to be constant in $[0, T]$. Put $B_t := e^{r\,t}$ so that $B = \{B_t : t \in [0, T]\}$ describes the price process of a zero–coupon bond.

- Let X_t denote the price of a risky asset (like a stock or an index) at $t \in [0, T]$. $X = \{X_t : t \in [0, T]\}$ describes the random price process of the risky asset whose analytical shape is unknown.

However, given suitable assumptions concerning the movement of the stochastic price, we may use CRR–trees for a numerical approximation in the following sense:

- Given an arbitrary $n \in \mathbb{N}$, consider an equidistant decomposition $Z_n := \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)}\}$ of the interval $[0, T]$: Let $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{k_n}^{(n)} = T$ so that $\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} (t_{j+1}^{(n)} - t_j^{(n)}) = 0$, where $\Delta_n := t_{j+1}^{(n)} - t_j^{(n)}$ is independent of j (for example, a bisection of the interval $[0, T]$).

- For every equidistant decomposition Z_n we consider the related CRR$^{(n)}$–tree.

- For every CRR$^{(n)}$–tree, the price $\pi_0^{(n)}(H)$ of an ECO $H = (X_T - K)^+$ is then given by

  $$\pi_0^{(n)}(H) = X_0 \cdot B(t_n^*, k_n; \tilde{p_n}) - \frac{K}{e^{rT}} \cdot B(t_n^*, k_n; q_n),$$

  where

  $$t_n^* = \min\{j \in \mathbb{N}_0 : X_0\, u_n^j\, d_n^{k_n - j} > K\},$$

  $r_n := e^{r\Delta_n} - 1$, $\tilde{p_n} = \frac{q_n\, u_n}{1+r_n}$, $q_n = \frac{(1+r_n)-d_n}{u_n - d_n}$ and $B(\tau, k_n; p) := \sum_{j=\tau}^{k_n} \binom{k_n}{j} p^t\, (1-p)^{k_n - j} = \mathbb{P}(\tau \le Z_n \le k_n)$ with binomially distributed $Z_n \sim \mathrm{BIN}(k_n; p)$ $(\tau \in \{0, 1, \ldots, k_n\}$ and $0 < p < 1)$.

- Notice that for every CRR$^{(n)}$–tree the sole "free" parameters are the values u_n and d_n only!

We choose the parameters u_n and d_n in the following sense:

Theorem 8 (Black–Scholes formula) *Consider an ECO $H = (X_T - K)^+$ with expiry T and strike price K, written on X. Let $\sigma > 0$ be a constant so that for all $n \in \mathbb{N}$ $r\sqrt{\Delta_n} < \sigma$. Put*

(i) $u_n := \exp(\sigma\sqrt{\Delta_n})$,

(ii) $d_n := \exp(-\sigma\sqrt{\Delta_n}) = \dfrac{1}{u_n}$.

Then there exists

$$\lim_{n\to\infty} \pi_0^{(n)}(H) =: \pi_0^{BS}(H),$$

where

$$\pi_0^{BS}(H) = X_0 \cdot \Phi(d_1(X_0, T)) - \frac{K}{e^{rT}} \cdot \Phi(d_2(X_0, T))$$

with

$$d_1(x,t) := \frac{\ln(\frac{x}{K}) + (r + \frac{\sigma^2}{2}\,t)}{\sigma\sqrt{t}}$$

and

$$d_2(x,t) := \frac{\ln(\frac{x}{K}) + (r - \frac{\sigma^2}{2}\,t)}{\sigma\sqrt{t}}.$$

σ is called the *volatility* of X. Notice that

$$\exp(\sigma\sqrt{\Delta}) \stackrel{!}{=} \exp(\sqrt{\mathrm{Var}(\sigma W_\Delta)}) \quad \forall\,\Delta \geq 0,$$

where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion. Hence, given the assumptions of the Black–Scholes model (in particular, constant volatility !), the price process of the risky asset X appears as a geometric Brownian motion in the risk neutral world:

$$X_t = X_0 \exp((r - \frac{\sigma^2}{2})\,t + \sigma W_t^*),$$

which is a solution of the following SDE:

$$dX_t = r\, X_t\, dt + \sigma\, X_t\, dW_t^*.$$

Let $X_j^{(n)}$ denote the price of the risky asset in the CRR$^{(n)}$–tree at time $t_j^{(n)}$. In particular: $X_{k_n}^{(n)} = X_T$. If we choose σ as above, we arrive at the following:

Remark 2 *Given the assumptions of Theorem 8, it follows that*

$$\lim_{n\to\infty} Var_{\boxed{q_n}}\Big(\ln\big(\frac{X_T}{X_0}\big)\Big) = \sigma^2\, T$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\boxed{q_n}}\Big(\ln\big(\frac{X_T}{X_0}\big)\Big) = \mu\, T,$$

where $\mu := r - \frac{\sigma^2}{2}$ *and* $q_n := \frac{(1+r_n)-d_n}{u_n-d_n}$ *denotes the EMM (in the CRR$^{(n)}$–tree).*