A few remarks on the pricing of contingent convertibles

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Some tight definition of CoCos (from `ft.com/lexicon`)

Contingent convertibles, also known as CoCo bonds, Cocos or contingent convertible notes, are slightly different to regular convertible bonds in that the likelihood of the bonds converting to equity is “contingent” on a specified event, such as the stock price of the company falling below a particular level for a certain period of time.

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CoCos are different to existing hybrids because they are designed to convert into shares if a pre-set trigger is breached in order to provide a shock boost to capital levels and reassure investors more generally.

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Hybrids, including CoCos, contain features of both debt and equity. They are intended to act as a cushion between senior bondholders and shareholders, who will suffer first if capital is lost.

Key elements of a CoCo

When designing a CoCo, there are mainly two elements that need to be mentioned: the *conversion trigger* (i. e., the specified event that implies partial conversion of the bond into shares or a write-down) and the *conversion ratio*, describing a predefined number of shares into which the bonds will be converted (cf. [5]).

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- Different types of trigger events can be defined including a market trigger and a regulatory trigger, and an accounting trigger;
- Different interests: shareholders, investors, CoCo issuers, regulators.

Market trigger

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- *Disadvantages*: Market could reflect distorted data - particularly during stressed periods; market manipulation cannot be excluded.

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- *Advantages*: Uncertainty of the entry time of the government's decision reduces market manipulation;
- *Disadvantages*: Negative signal in case of regulatory triggering causes regulators to shift their decision for conversion, leading to unpredictable implications due to the induced time delay.

Model assumption I

Let τ denote the *random* conversion time, S_t be the value of the firm's stock (respectively equity) at time $0 \leq t \leq T$ and F be the face value of the CoCo. Let n the (non-random) number of shares, handed over to the buyer of the CoCo in case of conversion.

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Model Assumption in [5]

Given that the trigger knocked in at $\tau \leq T$, then – at T – only the part $(1-\alpha) \cdot F$ would be paid as cash to the investor, for some $0 < \alpha \leq 1$, in addition to a partial coupon payment series, which would be cancelled at the arrival of τ. The remaining part αF would then be converted into nS_T.

Model assumption II

Model Assumption in [5] (ctd.)

We assume absence of arbitrage ($\rightsquigarrow$ NFLVR), implying the existence of at least one "risk neutral" measure $\mathbb{Q}$. Firstly, we are considering the standard Black-Merton-Scholes model only, implying that - under $\mathbb{Q}$ - the stock price process is modelled as (continuous) geometric Brownian motion:

$$S_t = S_0 \exp(X_t) = S_0 \exp\left(\nu t + \sigma W_t\right),$$

where $W \equiv (W_t)_{t \geq 0}$ is a standardised BM (under $\mathbb{Q}$), $S_0 > 0, \sigma > 0$, $r > 0$ ("risk-free rate"), $q > 0$ ("constant continuous dividend yield"), $\nu := r - q - \frac{1}{2}\sigma^2$, and $X_t := \nu t + \sigma W_t$ $(0 \leq t \leq T)$.

The payoff of a CoCo I

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Let $0 = t_0 < t_1 < t_2 < \ldots < t_M = T$, such that if $t_j < \tau(\omega) \leq t_{j+1}$ for some j, then only the (non-random) coupons $c_1, c_2, \ldots, c_j > 0$ would be paid to the investor of the CoCo.

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The payoff of a CoCo II

$$PO_{\mathrm{CC}}(T) = \sum_{i=1}^{M} c_i e^{r(T-t_i)} \mathbb{1}_{\{\tau > t_i\}} + \mathbb{1}_{\{\tau > T\}} F + \mathbb{1}_{\{\tau \leq T\}} \big(nS_T + (1-\alpha)F \big)$$

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&= \sum_{i=1}^{M} c_i e^{r(T-t_i)} \mathbb{1}_{\{\tau > t_i\}} + F \\
&+ \; n \Big((S_T - C_p)^{+} \mathbb{1}_{\{\tau \le T\}} - (C_p - S_T)^{+} \mathbb{1}_{\{\tau \le T\}} \Big) \, ,
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$$\sum_{i=1}^{M} c_i e^{r(T-t_i)} \mathbb{1}_{\{\tau > t_i\}} = \sum_{i=1}^{j^*-1} c_i e^{r(T-t_i)} ,$$

respectively

$$\sum_{i=1}^{M} c_i e^{r(T-t_i)} \mathbb{1}_{\{\tau \leq t_i\}} = \sum_{i=j^*}^{M} c_i e^{r(T-t_i)} .$$

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The payoff of a CoCo V

$$C_T := n\left((S_T - C_p)^+ \mathbb{1}_{\{\tau \le T\}} - (C_p - S_T)^+ \mathbb{1}_{\{\tau \le T\}}\right),$$

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$PO_{\mathrm{ZCC}}(T) := F + C_T$ (coinciding with the payoff of a ZCCoCo) .

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Thus, the discounted payoff (discounted to $t_0 = 0$) is given by

$$e^{-rT} PO_{\mathrm{CC}}(T) = e^{-rT}(A_T + B_T + C_T) = \tilde{A}_T + \tilde{B}_T + \tilde{C}_T\,,$$

where $\tilde{A}_T := e^{-rT} A_T$, $\tilde{B}_T := e^{-rT} B_T$ and $\tilde{C}_T := e^{-rT} C_T$.

Risk neutral pricing of the Coupon CoCo I

Firstly, to keep the machinery simple, consider the calculation of the “risk neutral” price under one fixed “risk neutral” measure $\mathbb{Q}$ at time $t = t_0 = 0$ only. Let $P_{\text{CC}}(0)$ denote this price. Then

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$$P_{\mathsf{CC}}(0) = \tilde{A}_T - \sum_{i=1}^{M} c_i \, e^{-r\, t_i} \, \mathbb{Q}(\tau \leq t_i) + \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T] \, .$$

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The calculation of $\mathbb{E}_{\mathbb{Q}}[\tilde{C}_T]$ and the default probability $\mathbb{Q}(\tau \leq t_i)$ requires the calculation of the distribution function of the random trigger time τ and the pricing of (path dependent) exotic options, such as barrier options (including a down-and-in barrier call option on the strike C_p (DIC), together with a down-and-in barrier call put option on the strike C_p (DIP)).

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$\nu := r - q - \frac{1}{2}\sigma^2$. Applying [4] - Prop. A.18.1 to our process $Y = y_0 + X$, where $y_0 := \ln\left(\frac{S_0}{H}\right)$, we immediately obtain the next part of the pricing formula in [5]:

Calculation of $\mathbb{Q}(\tau_H \leq t_i)$ II

Proposition

τ_H has an inverse Gaussian probability distribution under $\mathbb{Q}$, that is, for any $t > 0$,

$$\mathbb{Q}\big(\tau_H \leq t\big) = \Phi(z_1(t)) + \left(\frac{S_0}{H}\right)^{-2\nu/\sigma^2} \Phi(z_2(t)),$$

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where

$$z_j(t) := \frac{\ln\left(\frac{H}{S_0}\right) + (-1)^j \nu t}{\sigma\sqrt{t}} \qquad (j = 1, 2)\,.$$

Interrelation between the language of [5] and our notation

Let $i = 1, 2, \ldots, M$. Then:

$$x_{1i} = -z_1(t_i) + \sigma\sqrt{t_i} \text{ and } x_1 = x_{1M} = -z_1(T) + \sigma\sqrt{T}$$

and

$$y_{1i} = z_2(t_i) + \sigma\sqrt{t_i} \text{ and } y_1 = y_{1M} = z_2(T) + \sigma\sqrt{T}\,.$$

Moreover, we have:

$$r - q - \frac{\sigma^2}{2} =: \nu = (\lambda - 1)\sigma^2\,,$$

implying that in fact $2\nu/\sigma^2$ coincides with $2\lambda - 2$, where $\lambda := r - q + \frac{\sigma^2}{2}/\sigma^2$ (as defined in [5]).

Risk neutral pricing of the Coupon CoCo II

Next, we consider the calculation of the “risk neutral” price under the fixed “risk neutral” measure $\mathbb{Q}$ at time $t > t_0 = 0$, impying that we have to make use of conditional expectation operators.

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$$P_{\text{CC}}(t) = e^{rt}\left(\tilde{A}_T - \sum_{i=1}^{M} c_i\, e^{-r\, t_i}\, \mathbb{Q}\big(\tau_H \leq t_i | \mathcal{F}_t\big) + \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T | \mathcal{F}_t]\right).$$

Calculation of $\mathbb{Q}(\tau_H \leq t|\mathcal{F}_u)$

Corollary (conditional default probability)

For any $0 \leq u < t$ *on the event* $\{\tau_H > u\}$, *we have*

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where for $j = 1, 2$ and $0 \leq u < t$ the random variable $Z_j(u,t)$ is given as

$$Z_j(u,t) := \frac{-Y_u + (-1)^j \nu (t-u)}{\sigma\sqrt{t-u}} = \frac{\ln\left(\frac{H}{S_u}\right) + (-1)^j \nu(t-u)}{\sigma\sqrt{t-u}}.$$

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Calculation of $e^{rt}\mathbb{E}_{\mathbb{Q}}[\tilde{C}_T|\mathcal{F}_t]$ II

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A few tasks for research and consulting I

- Early exercise at (or soon after) a touch of the CoCo barrier

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau} PO_{\text{CC}}(\tau)\mathbb{1}_{\{\tau_H \leq \tau\}}\right] =?$$

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- CoCos and stochastic interest rates, respectively multi-curve framework ($\rightsquigarrow$ pricing of barrier options in the context of stochastic interest rates, respectively Overnight Index Swap (OIS) rates).

Time changes and Monroe's Theorem I

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions.

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A time change is a right-continuous non-decreasing $[0, \infty]$-valued process $(T_s)_{s \geq 0}$ such that T_s is a stopping time for any $s \geq 0$. The time change is called finite if for any $s \geq 0$ $T_s < \infty$ $\mathbb{P}$-almost surely.

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Theorem (Monroe (1978) $+$ Barndorff-Nielsen and Shiryaev (2010))

A càdlàg process $X = (X_s)_{s\geq 0}$ is a semimartingale if and only if there is a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t\geq 0}, \widehat{\mathbb{P}})$, an $\widehat{\mathbb{F}}$-adapted $\widehat{\mathbb{P}}$-Brownian motion $W = (W_s)_{s\geq 0}$ and a finite $\widehat{\mathbb{F}}$-time change $T \equiv (T_s)_{s\geq 0}$ such that the processes X and $W_T = (W_{T_s})$ have the same law.

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Time changes and Monroe's Theorem II

Theorem (Stochastic Exponential)

Let S be an $\mathbb{R}$-valued semimartingale such that $S_0 = 0$. Put $V_0 := 0$ and

$$V_t := \prod_{0<s\leq t} (1 + \Delta S_s) \cdot e^{-\Delta S_s} \quad (t > 0)\,.$$

Then the infinite product is a. s. absolutely convergent, and $V = (V_t)$ is an adapted purely discontinuous process which is of finite variation. Put

$$\mathcal{E}(S)_t := Z_t := \exp\left(S_t - \frac{1}{2}[S,S]_t^c\right) \cdot V_t\,.$$

Then $Z_0 = 1$, and $Z = (Z_t)$ is the unique semimartingale which is a solution of the following stochastic integral equation

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$$Z = 1 + \int Z_- \, dS\,, \text{ respectively } dZ = Z_- \, dS\,.$$

Time changes and Monroe's Theorem III

The mapping $S \mapsto \mathcal{E}(S) = Z$ can be inverted if both, Z and Z_- are $\mathbb{R}^*$-valued (outside of an evanescent set) [cf. e. g. Kallsen and Shiryaev (2002)].

Time changes and Monroe's Theorem III

The mapping $S \mapsto \mathcal{E}(S) = Z$ can be inverted if both, Z and Z_- are $\mathbb{R}^*$-valued (outside of an evanescent set) [cf. e. g. Kallsen and Shiryaev (2002)].

Theorem (Stochastic Logarithm)

Let

$$\mathcal{S}_0^\sim := \left\{ S \in \mathcal{S} \,\middle|\, \{\, \Delta S = -1\} \text{ is evanescent}, S_0 = 0 \right\}$$

and

$$\mathcal{S}_1^* := \left\{ Z \in \mathcal{S} \,\middle|\, \{ZZ_- = 0\} \text{ is evanescent},\ Z_0 = 1 \right\}.$$

Time changes and Monroe's Theorem IV

Theorem (Stochastic Logarithm - contd.)

Then the mapping

$$\begin{aligned} \mathcal{E} : \mathcal{S}_0^{\sim} &\overset{\simeq}{\longrightarrow} \mathcal{S}_1^{*} \\ S &\mapsto \mathcal{E}(S) \end{aligned}$$

is bijective, and its inverse is given by

$$\begin{aligned} \mathcal{L} : \mathcal{S}_1^{*} &\overset{\simeq}{\longrightarrow} \mathcal{S}_0^{\sim} \\ Z &\mapsto \int \frac{1}{Z_-} dZ - 1\,. \end{aligned}$$

Time changes and Monroe's Theorem V

Corollary

Let $Z = (Z_s)_{s \geq 0}$ *be a semimartingale. Then* $Z \in \mathcal{S}_1^*$ *if and only if there is a filtered probability space* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t \geq 0}, \widehat{\mathbb{P}})$*, an* $\widehat{\mathbb{F}}$*-adapted* $\widehat{\mathbb{P}}$*-Brownian motion* $W = (W_s)_{s \geq 0}$ *and a finite* $\widehat{\mathbb{F}}$*-time change* $T \equiv (T_s)_{s \geq 0}$ *such that the processes* Z *and* $\mathcal{E}\left(W_T\right) = \mathcal{E}(W)_T$ *have the same law.*

Time changes and Monroe's Theorem VI

Example (Escobar, Hieber and Scherer (2014))

Many well-known models can be represented as a time-changed Geometric Brownian motion including

- *stochastic volatility models: Heston, Stein & Stein, Hull-White, certain continuous limits of GARCH models;*
- *the Normal Inverse Gaussian (NIG) model;*
- *Sato models: for example extensions of the Variance Gamma (VG) model;*
- *The Ornstein-Uhlenbeck (OU) process.*

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Questions? Remarks?