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Notes and Comments

The numeraire portfolio in financial markets modeled by a multi-dimensional jump diffusion process

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1. Introduction and Summary

In continuous time $0 \le t \le T < \infty$, we study a financial market which is free of arbitrage opportunities but incomplete under the physical probability measure P. Thus one has several choices of equivalent martingale measures. In the present paper, the (unique) martingale measure P^* is studied which is defined by the concept of the numeraire portfolio. The choice of P^* can be justified by a change of numeraire in place of a change of measure.

In the market $1+d$ assets can be traded. One of them is the bank account (bond) and is described by the interest rate r; then $B_t := e^{rt}$ serves as a numeraire in the classical sense. The other d assets are called stocks and are described by the d-dimensional price process $\{S_t; 0 \le t \le T\}$. Besides $\{B_t\}$, we consider numeraires $\{V_t\}$ which are given by the positive value process of a self-financing dynamic portfolio with initial capital 1.

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We ask for the existence of a martingale measure P^* and a dynamic portfolio with positive value process $\{V_t^*\}$ where $V_0^* = 1$ such that

$$\frac{dP^*}{dP} \cdot \frac{1}{B_T} = \frac{1}{V_T^*}. \tag{1.1}$$

As a consequence we then obtain for any contingent claim H

$$E_{P^*}\Big[\frac{1}{B_T} \cdot H\Big] = E\Big[\frac{1}{V_T^*} \cdot H\Big], \tag{1.2}$$

i.e., we can replace the change of measure $P \hookrightarrow P^*$ by the change of numeraire $\{B_t\} \hookrightarrow \{V_t^*\}$.

It is known (see Long (1990), Becherer (2001), Korn and Schäl (1999), Schäl (2000)) that in many cases one can get a numeraire portfolio by maximizing the expected utility when using the log-utility. Here we are interested in optimal portfolio vectors which can be chosen from the *interior* of the set of admissible portfolio vectors and thus lead to a first order condition. The numeraire portfolio was further analyzed by Artzner (1997), Bajeux-Besnainou and Portait (1997), Johnson (1996), De Santis, Gerard & Ortu (2000). The present paper is related to Becherer (2001) where a general semimartingale model is treated. However, in the general model, the martingale condition or even a local martingale condition (used for the concept of a martingale measure) is too stringent to obtain a general existence result for a numeraire portfolio. Therefore Becherer used the weaker concept of a (generalized) numeraire portfolio whereas in this paper the existence of a numeraire portfolio in the original strict sense is proved.

The return processes of the price process will be jump-diffusion processes. For convenience, we restrict attention to the case of constant coefficients; then the jump-diffusion processes are Lévy processes. This has the advantage that we obtain a numeraire portfolio defined by a single (deterministic) portfolio vector $\vartheta^* \in \mathbf{R}^d$. Since the martingale property is a local one both in time and in space and since dynamic portfolios can also be defined locally both in time and in space, the results extend to more general jump-diffusion processes.

The paper is organized as follows. In Section 2 below a precise formulation of the market model is given. In Section 3 the structure of martingale measures and their connection to the numeraire portfolio is discussed. Then the existence result is presented in Section 4, while the proof is deferred to Section 5.

2. The model

We consider a financial market in continuous time $0 \leq t \leq T$ where T, $0 < T < \infty$, is the time-horizon. An investor can invest in a bank account and d stock accounts. For convenience and without loss of generality, the interest rate r of the bank account is assumed to be zero, i.e., $B_t = 1$, $t \geq 0$. Otherwise the analysis would be given in terms of discounted price processes. The stocks are described by the price process $\{S_t = (S_t^1, ..., S_t^d); 0 \leq t \leq T\}$ where S_t^k is the price of one share of stock k at time t.

We interpret a (deterministic or random) vector z as a column vector. Moreover "$\top$" will denote the transpose and $\langle z, y\rangle = z^\top y$ the inner product in $\mathbf{R}^d$.

It is convenient to write the price process as a stochastic exponential of the return process. The *return process* $\{R_t = (R_t^1, ..., R_t^d)^\top; 0 \leq t \leq T\}$ is defined by $dS_t^k = S_{t-}^k dR_t^k$, $1 \leq k \leq d$.

In our case the return process is driven by a d-dimensional compound Poisson process $\{Y_t\}$ and an m-dimensional Wiener process $\{W_t\}$. Then $\{Y_t\}$ can be defined by $Y_t = Z_1 + \cdots + Z_{N_t}$ where $\{N_t\}$ is a Poisson process with jump rate λ and $\{Z_n = (Z_n^1, \ldots, Z_n^d)^\top; n \geq 1\}$ is the sequence of iid jump sizes in $\mathbf{R}^d$. We set $Z := Z_1$ for a typical jump size and assume $1 + Z^k > 0$ a.s. $\forall k$.

Moreover there is an m-dimensional (standard) Wiener process $\{W_t = (W_t^1, \ldots, W_t^m)^\top; 0 \leq t \leq T\}$ which is independent of the compound Poisson process $\{Y_t\}$. Then the return process $\{R_t\}$ is assumed to be given by $R_t^k = (\sigma^k)^\top W_t + Y_t^k + a^k t$ where a^k is the kth element of the vector $a \in \mathbf{R}^d$ and $(\sigma^k)^\top \in \mathbf{R}^m$ is the kth row vector of the matrix $\sigma \in \mathbf{R}^{d\times m}$.

In this paper we consider stochastic exponentials $\{\mathcal{E}(X)_t\}$ of (càdlàg) semimartingales $\{X_t\}$ with jump times $\{T_n\}$ such that the number N_t of jumps in $(0, t]$ is finite and, without loss of generality, $X_0 = 0$. Then the general formula (see Protter (1995), Section II.8) simplifies to

$$\mathcal{E}(X)_t = \exp\{X_t^c - \frac{1}{2}[X^c, X^c]_t\} \cdot \prod_{n=1}^{N_t}(1 + \Delta X_{T_n})$$

where $\{X_t^c\}$ describes the continuous part of $\{X_t\}$ and $\{[X^c, X^c]_t\}$ its quadratic variation which here agrees with the conditional / predictable quadratic variation (see Protter (1995), p. 63). Thus the price process $\{S_t\}$ of the present model is given by

$$\begin{aligned} S_t^k &= S_0^k \cdot \mathcal{E}(R^k)_t \\ &= S_0^k \cdot \exp\left\{(\sigma^k)^\top W_t + a^k t - \frac{1}{2}\|\sigma^k\|^2 t\right\} \cdot \prod_{n=1}^{N_t}(1 + Z_n^k). \end{aligned} \tag{2.1}$$

The underlying return processes are Lévy processes. We allow for the case "$\sigma = 0$"; then the stock price process is a multidimensional compound Poisson process plus a linear term. In the same way we admit the case

"$\lambda = 0$" where we have a pure diffusion process. In particular, we do not need any assumption of nondegeneracy as, e.g., in Bardhan and Chao (1996).

A *trading strategy* $\{\phi_t\} = \{(\phi_t^1, \ldots, \phi_t^d)^\top; 0 \le t \le T\}$ is defined as a predictable $\mathbf{R}^d$-valued process where ϕ_t^k denotes the number of shares of stock k which the investor holds just before rebalancing the holdings at time t. In this paper, we consider self-financing strategies starting with an initial wealth $V_0 = 1$. Then the number of units of money in the bank account need not to be specified. This can be seen from the following formula for the value process $\{V_t\}$:

$$V_t = 1 + \int_0^t \phi_u^\top \cdot dS_u = 1 + \int_0^t \langle \phi_u, dS_u \rangle.$$

We consider strategies ϕ with $V_{t-}^\phi > 0$ a.s. Then we can write

$$\begin{aligned} dV_t = \langle \phi_t, dS_t \rangle &= \sum_{k=1}^d \phi_t^k dS_t^k \\ &= \sum_{k=1}^d V_{t-}(\phi_t^k S_{t-}^k / V_{t-}) dS_t^k / S_{t-}^k \\ &= \sum_{k=1}^d V_{t-} \pi_t^k dR_t^k, \end{aligned}$$

where $\pi_t^k = \phi_t^k S_{t-}^k / V_{t-}$ denotes the proportion of the value of the shares of stock k which the investor holds just before time t and $\{\pi_t\} = \{(\pi_t^1, \ldots, \pi_t^d)^\top; 0 \le t \le T\}$ is called a *portfolio process*. In this paper as in many other applications where the return process is a Lévy process, it turns out that we can restrict attention to the case where the portfolio process is described by a single *portfolio vector* $\vartheta \in \mathbf{R}^d$ according to $\pi_t = \vartheta$ for all t, $0 \le t \le T$. In that case we have $dV_t = V_{t-} d(\langle \vartheta, R_t \rangle)$ where $\langle \vartheta, R_t \rangle = \langle \vartheta, \sigma W_t \rangle + \langle \vartheta, Y_t \rangle + \langle \vartheta, a \rangle t$.

Thus we can define the value process $\{V_t^\vartheta\}$ for the portfolio vector ϑ by $V_t^\vartheta = \mathcal{E}(\langle \vartheta, R \rangle)_t$, i.e.,

$$\begin{aligned} V_t^\vartheta = \exp &\left\{ \langle \vartheta, \sigma W_t \rangle + \langle \vartheta, a \rangle t - \frac{1}{2} \|\sigma^\top \vartheta\|^2 t \right\} \\ &\times \prod_{n=1}^{N_t} (1 + \langle \vartheta, Z_n \rangle) \end{aligned} \tag{2.2}$$

since $\langle \vartheta, \sigma W_t \rangle = (\sigma^\top \vartheta)^\top W_t$, where V_t^ϑ is well-defined for any $\vartheta \in \mathbf{R}^d$.

3. Martingale measures and the numeraire portfolio

We need to generalize the classical concept of a martingale measure.

Definition. *Suppose that the value process $\{V_t^\vartheta\}$ for a portfolio vector ϑ is positive a.s. Then the probability measure Q is called a* **martingale measure for the numeraire** $\{V_t^\vartheta\}$ *if and only if*

$$\{S_t^k / V_t^\vartheta\},\ 1 \le k \le d, \quad \textit{and} \quad \{B_t / V_t^\vartheta\} \quad \textit{are martingales under} \quad Q. \tag{3.1}$$

Here the price process $\{B_t\}$ (where $B_t = 1$ by assumption) describing the bank account plays the same rôle as the price process of any other stock. We remark that $\{V_t^\vartheta\} = \{B_t\}$ for $\vartheta = 0$.

Definition. *The portfolio vector ϑ^* is called a* **numeraire portfolio** *if and only if the (physical / real world) probability P is a martingale measure for the numeraire $\{V_t^{\vartheta^*}\}$.*

Becherer (2001) uses the same definition of a numeraire portfolio except that the weaker notion of a supermartingale (in place of a martingale) is used in (3.1). In the situation of (3.1), $L_t := 1/V_t^{\vartheta^*}$ is a positive martingale under P with $L_0 = 1$ and hence the density process of a probability measure P^* with $dP^*/dP = L_T = 1/V_T^{\vartheta^*}$. Then (3.1) implies that $\{L_t \cdot S_t^k\}$ are martingales under P which means that $\{S_t^k\}$ are martingales under P^*, $1 \le k \le d$ (see Protter (1995), Lemma, p. 109). Thus P^* is a martingale measure (in the classical sense) which is defined by the numeraire $\{V_t^{\vartheta^*}\}$ corresponding to the numeraire portfolio ϑ^*. It is known that P^* is unique, i.e., there exists at most one martingale measure defined by a numeraire portfolio (see Becherer (2001), Korn and Schäl (1999)). Then we have the situations (1.1) and (1.2) with $V_t^* = V_t^{\vartheta^*}$. Our problem can now be formulated in the following way:

Find $\vartheta^* \in \mathbf{R}^d$ such that $\{L_t = 1/V_t^{\vartheta^*}\}$ is a martingale under P and $\{L_t \cdot S_t^k\}$ are martingales under P for $1 \le k \le d$.

In the next part we first discuss density processes with homogeneous dynamics which induce martingale measures. We know the form of possible candidates for density processes (see Björk, Kabanov and Runggaldier (1997), Shirakawa (1990), Bardhan and Chao (1996)) which are defined by a vector $\psi \in \mathbf{R}^m$ and a function $\alpha : \mathbf{R}^d \mapsto \mathbf{R}$, where we write $\bar\alpha := E[\alpha(Z)]$:

$$\begin{aligned} L_t &= L_t^{\alpha,\psi} \\ &= \mathcal{E}(\Lambda^{\alpha,\psi})_t, \quad \text{where } \alpha(Z_n) > 0 \text{ a.s., } \bar\alpha < \infty, \end{aligned} \tag{3.2}$$

$$\Lambda_t^{\alpha,\psi} = -\psi^\top W_t + \sum_{n=1}^{N_t} [\alpha(Z_n) - 1] - \lambda(\bar\alpha - 1)t. \tag{3.3}$$

Here $\left\{\sum_{n=1}^{N_t}[\alpha(Z_n)-1]-\lambda(\bar{\alpha}-1)t\right\}$ is a martingale under P and thus a compensated compound Poisson process. As a consequence, as the stochastic exponential of a martingale, $\{L_t^{\alpha,\psi}\}$ is again a martingale under P. From (3.2) and (3.3) we see that

$$L_t^{\alpha,\psi} = \exp\left\{-\psi^\top W_t - \frac{1}{2}\|\psi\|^2 t - \lambda(\bar{\alpha}-1)t\right\} \cdot \prod_{n=1}^{N_t} \alpha(Z_n). \tag{3.4}$$

Now we want to study the following problem: when is $\{L_t^{\alpha,\psi} \cdot S_t^k\}$ a martingale under P? By (2.1) and (3.4),

$$\begin{aligned} L_t^{\alpha,\psi} \cdot S_t^k = S_0^k \cdot \exp\Bigg\{&(\sigma^k-\psi)^\top W_t \\ &+ \left[a^k - \frac{1}{2}\|\psi\|^2 - \lambda(\bar{\alpha}-1) - \frac{1}{2}\|\sigma^k\|^2\right] \cdot t\Bigg\} \\ &\times \prod_{n=1}^{N_t} \beta^k(Z_n), \end{aligned}$$

where we set

$$\beta^k(z) := (1+z^k)\cdot\alpha(z)\,, \quad \bar{\beta}^k := E[\beta^k(Z)]\,.$$

If $\bar{\beta}^k < \infty$, then

$$\begin{aligned} L_t^{\alpha,\psi} \cdot S_t^k = S_0^k \cdot \exp&\left\{(\sigma^k-\psi)^\top W_t - \frac{1}{2}\|\sigma^k-\psi\|^2 \cdot t - \lambda(\bar{\beta}^k-1)t\right\} \\ &\times \prod_{n=1}^{N_t} \beta^k(Z_n) \\ &\times \exp\Bigg\{\Bigg[a^k - \frac{1}{2}\|\psi\|^2 - \lambda(\bar{\alpha}-1) \\ &\qquad - \frac{1}{2}\|\sigma^k\|^2 + \frac{1}{2}\|\sigma^k-\psi\|^2 + \lambda(\bar{\beta}^k-1)\Bigg] \cdot t\Bigg\}. \end{aligned}$$

The first factor has the form (3.4) and is thus a martingale. The last factor is equal to 1 if

$$a^k - \frac{1}{2}\|\psi\|^2 - \lambda(\bar{\alpha}-1) - \frac{1}{2}\|\sigma^k\|^2 + \frac{1}{2}\|\sigma^k-\psi\|^2 + \lambda(\bar{\beta}^k-1) = 0,$$

i.e., if

$$a^k - (\sigma^k)^\top \psi + \lambda(\bar{\beta}^k - \bar{\alpha}) = 0,$$

where

$$\bar{\beta}^k - \bar{\alpha} = E[Z^k \cdot \alpha(Z)].$$

Thus we proved the following facts.

Lemma 3.1. *Let be $\psi \in \mathbf{R}^m$ and, for $\alpha : \mathbf{R}^d \mapsto \mathbf{R}$, assume that $\alpha(Z) > 0$ a.s., $E[(1+\|Z\|)\cdot\alpha(Z)] < \infty$. Then the following properties are equivalent:*

(i) *the following vector equation holds in $\mathbf{R}^d$:*

$$a - \sigma\psi + \lambda E[\alpha(Z)Z] = 0; \tag{3.5}$$

(ii) *the probability measure $P^{\alpha,\psi}$ with density $\frac{dP^{\alpha,\psi}}{dP} = L_T^{\alpha,\psi}$ and density process $\{L_t^{\alpha,\psi}\}$ is a martingale measure (in the classical sense);*

(iii) *$\{L_t^{\alpha,\psi} \cdot S_t^k\}$ is a martingale under P for $1 \le k \le d$.*

In the final step, we ask for the existence of α, ψ, ϑ such that $L_t^{\alpha,\psi} = 1/V_t^\vartheta$. We start with a computation of $1/V_t^\vartheta$ by use of (2.2) and obtain $1/V_t^\vartheta = \mathcal{E}(\Delta)_t$ with

$$\Delta_t = -(\sigma^\top\vartheta)^\top W_t - \langle\vartheta, a\rangle t + \|\sigma^\top\vartheta\|^2 t + \sum_{n=1}^{N_t}\Big[\frac{1}{1+\langle\vartheta, Z_n\rangle} - 1\Big]. \tag{3.6}$$

By (3.2), (3.3) and (3.6) we reach our goal $L_t^{\alpha,\psi} = 1/V_t^\vartheta$ if $\Lambda_t^{\alpha,\psi} = \Delta_t$, i.e., if

$$\begin{aligned} &-\psi^\top W_t + \sum_{n=1}^{N_t}\Big[\alpha(Z_n) - 1\Big] - \lambda(\bar{\alpha} - 1)t \\ &\quad = -(\sigma^\top\vartheta)^\top W_t + \sum_{n=1}^{N_t}\Big[\frac{1}{1+\langle\vartheta, Z_n\rangle} - 1\Big] \\ &\qquad - \lambda E\Big[\frac{1}{1+\langle\vartheta, Z_n\rangle} - 1\Big]\cdot t \\ &\qquad + \{\lambda E\Big[\frac{1}{1+\langle\vartheta, Z_n\rangle} - 1\Big] - \langle\vartheta, a\rangle + \|\sigma^\top\vartheta\|^2\}\cdot t. \end{aligned} \tag{3.7}$$

Obviously (3.7) is satisfied under the following conditions:

$$\psi = \sigma^\top \vartheta, \tag{3.8a}$$

$$\alpha(z) = \frac{1}{1 + \langle \vartheta, z \rangle} \tag{3.8b}$$

$$\langle \vartheta, a \rangle = \|\sigma^\top \vartheta\|^2 + \lambda E\Big[\frac{1}{1 + \langle \vartheta, Z \rangle} - 1\Big]. \tag{3.8c}$$

Under (3.8a) and (3.8b), the condition (3.5) for a martingale measure can be written as the vector equation

$$a - \sigma\sigma^\top \vartheta + \lambda E\Big[\frac{1}{1 + \langle \vartheta, Z \rangle} Z\Big] = 0. \tag{3.9}$$

Fortunately, we need not consider (3.8c) which is just the condition that $\{\Delta_t\}$, and thus $\{1/V_t^\vartheta\} = \mathcal{E}(\Delta)$, be a martingale. In fact we have the following relation.

Lemma 3.2. *The condition* (3.9) *implies the condition* (3.8c).

Proof. From (3.9) we obtain

$$\begin{aligned}\langle \vartheta, a \rangle &= \langle \vartheta, \sigma\sigma^\top \vartheta \rangle - \lambda E\Big[\frac{\langle \vartheta, Z \rangle}{1 + \langle \vartheta, Z \rangle}\Big] \\ &= \|\sigma^\top \vartheta\|^2 + \lambda E\Big[\frac{1}{1 + \langle \vartheta, Z \rangle} - 1\Big].\end{aligned}$$

□

Finally we can summarize the considerations above in the following result.

Proposition 3.3. *If, for some* $\vartheta \in \mathbf{R}^d$*, one has* $0 < 1 + \langle \vartheta, Z \rangle$ *a.s. and if* α *and* ψ *are defined by* (3.8a) *and* (3.8b)*, then condition* (3.9) *implies that* $P^{\alpha,\psi}$ *[defined as in Lemma* 3.1 (ii)*] is a martingale measure with density process* $\{L_t^{\alpha,\psi}\} = \{1/V_t^\vartheta\}$ *and* ϑ *thus is a numeraire portfolio.*

4. Existence result

The existence of a numeraire portfolio implies the existence of a martingale measure (see (1.2)). Thus we need a no-arbitrage condition. Now, from Lemma 3.1 we obtain the following condition (NC) equivalent to the existence of a martingale measure of the form $P^{\alpha,\psi}$:

$$a - \sigma\psi + \lambda E[\alpha(Z)Z] = 0 \ \text{ for } \psi \in \mathbf{R}^m \text{ and } \alpha : \mathbf{R}^d \mapsto \mathbf{R}$$

$$\text{such that } \alpha(Z) > 0 \text{ a.s. and } E[(1 + \|Z\|) \cdot \alpha(Z)] < \infty. \tag{NC}$$

The condition (NC) is also necessary for the existence of a numeraire portfolio defined by a portfolio vector ϑ. In fact, in the latter case $\{1/V_t^{\vartheta}\}$ is a martingale and thus condition (3.8c) is satisfied. But then we see from (3.7) that $\Lambda_t^{\alpha,\psi} = \Delta_t$ with α and ψ as in (3.8a) and (3.8b); thus, $L_t^{\alpha,\psi} = 1/V_t^{\vartheta}$ and a martingale measure of the form $P^{\alpha,\psi}$ exists.

It can be shown (see Schäl and Szimayer (2001)) by use of a representation theorem for density processes (see Björk, Kabanov and Runggaldier (1997)) that condition (NC) is also necessary for, and thus equivalent to, the existence of an arbitrary equivalent martingale measure. Obviously, (NC) is satisfied if the diffusion part is nondegenerate in the sense that the rank of σ is d. We give a geometric interpretation of condition (NC) at the end of this section. Set supp(Z) for the support of Z and

$$\Theta := \{\vartheta \in \mathbf{R}^d; \quad 0 < 1 + \langle \vartheta, \zeta \rangle \quad \forall \zeta \in \mathrm{supp}(Z)\}.$$

For $\vartheta \in \Theta$ we thus have, according to (2.2), $V_t^{\vartheta} > 0$ a.s., and $\ln(V_t^{\vartheta})$ is defined a.s. as

$$\ln(V_t^{\vartheta}) = \langle \vartheta, \sigma W_t \rangle + \langle \vartheta, a \rangle t - \frac{1}{2}\|\sigma^\top \vartheta\|^2 \cdot t + \sum_{n=1}^{N_t} \ln(1 + \langle \vartheta, Z_n \rangle).$$

As mentioned in Section 1, one can get a numeraire portfolio by maximizing the expected utility when using the log-utility. In our case one then should consider

$$\begin{aligned} E[\ln(V_t^{\vartheta})] = J(\vartheta) \cdot t &:= \langle \vartheta, a \rangle \cdot t \\ &= -\frac{1}{2}\|\sigma^\top \cdot \vartheta\|^2 \cdot t + \lambda t \cdot E[\ln(1 + \langle \vartheta, Z \rangle)]. \end{aligned}$$

Now the first order condition $\frac{\partial}{\partial^k} J(\vartheta) = 0$ leads to

$$a^k - (\sigma^k)^\top \sigma^\top \vartheta + \lambda E\Big[\frac{1}{1 + \langle \vartheta, Z \rangle} Z^k\Big] = 0, \;\; 1 \le k \le d,$$

which is just (3.9). A similar formula was derived by Goll and Kallsen (2000), Example 4.3, and Kallsen (2000), Theorem 3.1. Thus we have to look for $\vartheta^* \in \Theta^o$ which is a maximizer of $J(\vartheta)$. Let $\mathcal{L}$ be the smallest linear space in $\mathbf{R}^d$ such that $Z \in \mathcal{L}$ a.s. From (NC) we conclude that $a = \sigma\psi - \lambda E[\alpha(Z)Z]$ for some $\psi \in \mathbf{R}^d$ and some α with $\alpha(Z) > 0$ a.s., $\bar{\alpha} < \infty$, $\bar{\beta}^k < \infty$, $1 \le k \le d$. Then

$$\begin{aligned} J(\vartheta) = \psi^\top \sigma^\top \vartheta &- \frac{1}{2}\|\sigma^\top \vartheta\|^2 \\ &+ \lambda \, E[\ln(1 + \langle \vartheta, Z \rangle) - \alpha(Z)\langle \vartheta, Z \rangle]. \end{aligned} \tag{4.1}$$

For the existence result, we restrict attention to the case in which Z takes on only finitely many values ζ_i, $1 \le i \le n$. Then it is clear that Θ is open. Now we can formulate our main result.

Theorem 4.1. *If the range of Z is finite and the condition* (NC) *holds, then there exists a vector $\vartheta^* \in \Theta$ satisfying* (3.9) *which thus defines a numeraire portfolio.*

The proof is given in the next section. Now we want to discuss the condition (NC). In the case in which the range of Z is finite, Shirakawa (1990) proved that condition (NC) is already satisfied only if there are no arbitrage opportunities. Now, let $\mathcal{H}$ be the smallest affine hyperplane containing $\text{supp}(Z)$ and thus the convex hull $\text{conv}(\text{supp}(Z))$. A point $\zeta^o \in \mathcal{H}$ is an *interior point* of $\text{conv}(\text{supp}(Z))$ *relative to* $\mathcal{H}$ if there exists a ball $B(\zeta^o, \epsilon)$ with radius $\epsilon > 0$ around ζ^o such that $B(\zeta^o, \epsilon) \cap \mathcal{H} \subset \text{conv}(\text{supp}(Z))$. Then we have the following result.

Proposition 4.2. *The condition* (NC) *is equivalent to the following condition:*

$$\begin{aligned} & a + c\,\zeta^o = \sigma\psi \;\; \textit{for some} \;\; \psi \in \mathbf{R}^m, \;\; \textit{some} \;\; c > 0, \;\; \textit{and some} \\ & \quad \textit{interior point} \;\; \zeta^o \;\; \textit{of} \;\; \text{conv}(\text{supp}(Z)) \;\; \textit{relative to} \;\; \mathcal{H}. \qquad \text{(NC*)} \end{aligned}$$

Proof. Assume (NC); then $\zeta^o := \frac{1}{E[\alpha(Z)]} E[\alpha(Z)Z]$ is obviously contained in $\text{conv}(\text{supp}(Z))$. By using a separating hyperplane, it is easy to see that ζ^o is even an interior point of $\text{conv}(\text{supp}(Z))$ relative to $\mathcal{H}$ and (NC*) condition with $c := \lambda E[\alpha(Z)]$. Now assume (NC*). Then 0 is an interior point of $\text{conv}(\text{supp}(Z - \zeta^o))$ relative to $\mathcal{H} - \zeta^o$.

Now we consider a one-period market model on $(\Omega, \sigma(Z), P)$, where $Z - \zeta^o$ is the difference ΔS of the price process. From the fundamental theorem (see Jacod and Shiryaev (1998), Theorem 3) we know that there is a density $\ell(Z) > 0$ a.s. with $E[\ell(Z)] = 1$ and $E[\ell(Z) \cdot (Z - \zeta^o)] = 0$, i.e., $0 = a - \sigma \cdot \psi + c \cdot \zeta^o = a - \sigma\psi + E[c \cdot \ell(Z) \cdot Z]$. Now we can set $\alpha = c \cdot \ell/\lambda$. □

In the pure diffusion case where $\lambda = 0$ the condition (NC) is well-known and θ is called the vector of the *risk premiums*. A consequence of Theorem 4.1 is the existence of a martingale measure P^* defined by a numeraire portfolio under condition (NC). For this diffusion case, the martingale measure P^* coincides with the minimal martingale measure (see Korn (1998), Becherer (2001)). In the model of the present paper, the so-called mean-variance trade off is deterministic. Thus the minimal martingale measure $\hat{P}$ coincides with the variance optimal martingale measure $\tilde{P}$ (see Schweizer (1996)).

The assumption of a finite range of Z is far from necessary. Indeed, in a model with only one stock (i.e., $d = 1$) and an infinite range of Z, it is easy to give conditions for the existence of the numeraire portfolio.

5. Proofs

Without loss of generality we assume that $\{\zeta_i; 1 \le i \le n\}$ is both supp(Z) and the range of Z. Let Γ be the orthogonal projection on $\mathcal{L}$. Then the linear space $\mathcal{L}$ is generated by the vectors ζ_i and

$$\langle \vartheta, \zeta_i \rangle = \langle \Gamma\vartheta, \zeta_i \rangle \forall i \text{ and hence } \langle \vartheta, Z \rangle = \langle \Gamma\vartheta, Z \rangle. \tag{5.1}$$

In view of (4.1) and (5.1), J can be written in the form

$$\begin{aligned} J(\vartheta) &= \hat{J}(\sigma^\top \vartheta, \Gamma\vartheta) \\ &= \psi^\top \sigma^\top \vartheta - \frac{1}{2}\|\sigma^\top \vartheta\|^2 \\ &\quad + \lambda E[\ln(1 + \langle \Gamma\vartheta, Z \rangle) - \alpha(Z)\langle \Gamma\vartheta, Z \rangle], \end{aligned}$$

where $\hat{J}(\xi, \eta) := \psi^\top \xi - \frac{1}{2}\|\xi\|^2 + \lambda E[\ln(1 + \langle \eta, Z \rangle) - \alpha(Z)\langle \eta, Z \rangle]$ is defined on the linear space

$$\hat{K} := \{(\sigma^\top \vartheta, \Gamma\vartheta) \in \mathbf{R}^{m+d}; \vartheta \in \mathbf{R}^d\}, \tag{5.2}$$

where $\ln x := -\infty$ for $x \le 0$. Now we construct a compact subset K of $\hat{K}$ such that $\hat{J} < 0$ on $\hat{K} \setminus K$. Since $0 \in \Theta$ and $J(0) = \hat{J}(0) = 0$, we know that $\hat{J}$ attains its maximum at some $(\sigma^\top \vartheta^*, \Gamma\vartheta^*)$ in K. Moreover it will become clear that $\vartheta^* \in \Theta$. Then ϑ^* is the required maximizer of J. We need the following lemma.

Lemma 5.1. *For $0 < M_i < \infty$, $0 < M < \infty$ the following set K is compact:*

$$\begin{aligned} K := \{&(\sigma^\top \vartheta, \Gamma\vartheta) \in \mathbf{R}^{m+d}; \vartheta \in \mathbf{R}^d, \\ &\|\sigma^\top \vartheta\| \le M, 1/M_i \le 1 + \langle \Gamma\vartheta, \zeta_i \rangle \le M_i, \forall i\}. \end{aligned} \tag{5.3}$$

Proof. Since $\hat{K}$ is a linear set, it is closed. Thus K is certainly closed. Suppose that K is unbounded, i.e., K contains a sequence $\{(\sigma^\top \vartheta_n, \Gamma\vartheta_n)\}$ such that $\|\Gamma\vartheta_n\| \to \infty$.

Set $\eta_n := (1/\|\Gamma\vartheta_n\|)\Gamma\vartheta_n$; then η_n lies on the compact sphere. Without loss of generality we may assume that $\eta_n \to \eta$ for some $\eta \in \mathcal{L}$, $\eta \ne 0$. We now have $0 \le (1/\|\Gamma\vartheta_n\|) + \langle \eta_n, \zeta_i \rangle \le M_i/\|\Gamma\vartheta_n\| \quad \forall i$ which implies that $0 \le \langle \eta, \zeta_i \rangle \le 0 \quad \forall i$ and hence $\eta \perp \mathcal{L}$ which is a contradiction. □

Proof of Theorem 4.1. We set $p_i := P[Z = \zeta_i] > 0$ and

$$w(\xi) := \psi^\top \xi - \tfrac{1}{2}\|\xi\|^2,\ \xi \in \mathbf{R}^m,$$
$$g_i(y) := p_i[\ln y - \alpha(\zeta_i)y],\ y > 0.$$

Then we obtain

$$\begin{aligned} J(\vartheta) &= w(\sigma^\top\vartheta) + \lambda \sum_i p_i[\ln(1 + \langle \Gamma\vartheta, \zeta_i\rangle) - \alpha(\zeta_i)\langle \Gamma\vartheta, \zeta_i\rangle] \\ &= w(\sigma^\top\vartheta) + \lambda \sum_i p_i[\ln(1 + \langle \Gamma\vartheta, \zeta_i\rangle) \\ &\quad - \alpha(\zeta_i)(1 + \langle \Gamma\vartheta, \zeta_i\rangle)] + \lambda \sum_i p_i \alpha(\zeta_i) \\ &= w(\sigma^\top\vartheta) + \lambda \sum_i g_i(1 + \langle \Gamma\vartheta, \zeta_i\rangle) + \lambda\bar{\alpha} = \hat{J}(\sigma^\top\vartheta, \Gamma\vartheta). \end{aligned}$$

We have

$$w(\xi) \le m \text{ for } \xi \in \mathbf{R}^d \text{ and some } m > 0,$$
$$w(\xi) \to -\infty \text{ as } \|\xi\| \to \infty; \tag{5.4}$$
$$g_i(0) = -\infty = g_i(\infty) \text{ and } g_i(y) \le m_i \text{ for}$$
$$0 < y < \infty \text{ and some } m_i > 0. \tag{5.5}$$

We know that for $\hat{K}$ as in (5.2):

$$\sup\{J(\vartheta);\ \vartheta \in \Theta\} = \sup\{\hat{J}(\xi, \eta);\ (\xi, \eta) \in \hat{K}\}.$$

First we study the case $\lambda > 0$.

By (5.4) and (5.5) we can choose the numbers M and M_i in the definition (5.3) of K such that

$$w(\xi) \le -\lambda \textstyle\sum_{i\in I} m_i - \lambda\bar{\alpha} - 1 \text{ for } \|\xi\| > M,$$
$$g_i(y) \le -\textstyle\sum_{j\in I} m_j - \bar{\alpha} - m/\lambda \text{ for } y > M_i \text{ and for } 0 < y < 1/M_i.$$

Thus, for $\|\xi\| > M$, $0 < y < \infty$, we obtain

$$\begin{aligned} &w(\xi) + \lambda \sum_i g_i(y) + \lambda\bar{\alpha} \\ &\quad \le -\lambda \sum_i m_i - \lambda\bar{\alpha} - 1 \\ &\qquad + \lambda \sum_i = m_i + \lambda\bar{\alpha} = -1 < 0. \end{aligned}$$

Similarly, if for some j we have $y > M_j$ or $0 < y < 1/M_j$, then, for $\xi \in \mathbf{R}^d$, we obtain

$$\begin{aligned} w(\xi) + \lambda \sum_i g_i(y) + \lambda\bar{\alpha} &= w(\xi) + \lambda\, g_j(y) + \lambda \sum_{i \neq j} g_i(y) + \lambda\bar{\alpha} \\ &\le m - \lambda \sum_i m_i - \lambda\bar{\alpha} - m + \lambda \sum_{i \neq j} m_i + \lambda\bar{\alpha} \\ &= -\lambda\, m_j < 0. \end{aligned}$$

Hence we know that

$$J(\vartheta) = w(\sigma^\top \vartheta) + \lambda \sum_j g_j(1 + \langle \Gamma\vartheta, \zeta_j \rangle) + \lambda\bar{\alpha} < 0$$

for $\|\sigma^\top \vartheta\| > M$ or for $\Gamma\vartheta \notin \cap_i \{\vartheta' \in \mathcal{L}\,;\, 1/M_i \le 1 + \langle \vartheta', \zeta_i \rangle \le M_i\}$.

Since $J(0) = 0$, we conclude that $\sup\{\hat{J}(\xi, \eta);\, (\xi, \eta) \in \hat{K}\}$ is attained on the compact set K. Thus we know that $\sup_\vartheta J(\vartheta)$ is attained at some ϑ^* where $(\sigma^\top \vartheta^*, \Gamma\vartheta^*) \in K$; in particular,

$$1/M_i \le 1 + \langle \vartheta^*, \zeta_i \rangle \le M_i \quad \forall i \text{ and thus } \vartheta^* \in \Theta.$$

As we saw above this implies that ϑ^* defines a numeraire portfolio.

The proof for the case $\lambda = 0$ is similar.

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