Mathematical Aspects of Credit Portfolio Management

Talk at the University of Konstanz

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Agenda

- Credit portfolio management (CPM) in a nutshell
- Credit risk in probability space
- Applications and examples
- Concluding remarks

• This talk represents the opinion of the author and does not necessarily reflect the opinion of Credit Suisse.
• Download of papers and background material at www.christian-bluhm.net

Agenda

- **Credit portfolio management (CPM) in a nutshell**
- Credit risk in probability space
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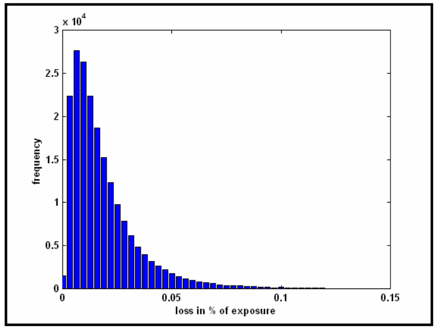
Credit portfolio management in a nutshell

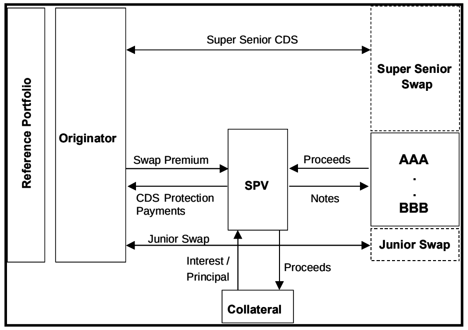
Credit portfolio management is a core competence of modern sophisticated banks

Measurement of risks and performance



Update

Management of credit risk (buy & sell)



Client and market orientation

Business-challenged

Innovative & up-to-date

Main task: Optimization of the risk/return profile of the bank‘s credit portfolio

Main lever of implementation: Consequent and up-to-date optimization of measurement and steering instruments at single-name and portfolio level

The central role of credit risk modeling

Application of credit risk models appear in three key areas

Credit risk management

- Measurement of the credit risk of clients, segments, and the total credit portfolio of the bank
- Limit setting, e.g., on single obligors, segments, businesses, etc.
- Economic and regulatory capital measurement and management
- Measurement and transparency regarding performance on risk capital
- Capital allocation, etc.

Credit portfolio management

- Active management and steering of the credit portfolio (includes RAP*)
- Active engagement in long and short positions in credit risks, hereby optimizing the risk/return profile of the credit portfolio (includes: hedging, securitizations, etc.)
- Improvement of capital velocity and capital efficiency
- Reduction of P&L volatility, etc.

Front office needs

- Sophistication credit risk pricing evaluations and related issues
- Structuring of credit transactions (e.g., in the bank's CDO group)
- Modeling of credit risky instruments
- Tailor-made credit products for clients of the bank, etc.

* RAP: risk-adjusted pricing/determination of credit risk premium

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Credit risk in probability space – basic modeling

$(\Omega, \mathcal{F}, \mathbb{P})$ a (suitable) probability space

Credit portfolio with names $i = 1, ..., m$ given

Bernoulli variable approach (default indicators):

$\mathbf{1}_{D_i^{(t)}}(\omega) = 1 \iff$ client i defaults up to time t in scenario $\omega \in \Omega$

Default event: $D_i^{(t)} = \{\omega \in \Omega : \mathbf{1}_{D_i^{(t)}}(\omega) = 1\} \in \mathcal{F}$

t-year default probability: $\text{PD}_i^{(t)} = p_i^{(t)} = \mathbb{P}[\mathbf{1}_{D_i^{(t)}} = 1] = \mathbb{P}(D_i^{(t)})$

Default time approach (survival times):

$\tau_i(\omega) < t \iff$ client i defaults in scenario $\omega \in \Omega$ before time t

Default/survival time: $\tau_i : (\Omega, \mathcal{F}) \rightarrow [0, \infty)$ measurable and continuous

t-year default probability: $\text{PD}_i^{(t)} = p_i^{(t)} = \mathbb{P}[\tau_i < t]$

Link between default indicator and default time: $D_i^{(t)} = \{\tau_i < t\}$

Credit risk in probability space – Bernoulli mixtures (1/2)

Bernoulli mixture model:

For $\delta_1, ..., \delta_m \in \{0, 1\}$ and $t > 0$

$$\mathbb{P}[\mathbf{1}_{D_1^{(t)}} = \delta_1, ..., \mathbf{1}_{D_m^{(t)}} = \delta_m] =$$

$$= \int_{[0,1]^m} \prod_{i=1}^{m} (p_i^{(t)})^{\delta_i}(1 - p_i^{(t)})^{1-\delta_i} \mathrm{d}\mathbb{F}^{(t)}(p_1^{(t)}, ..., p_m^{(t)})$$

quantifies the probability of joint defaults w.r.t. the time horizon t

$\mathbb{F}^{(t)}$ is called a mixture distribution, randomly drawing t-year PDs

Best-known market example:

$$\mathbb{P}\left[\sum_{i=1}^{m} \mathbf{1}_{D_i^{(t)}} = k\right] = \binom{m}{k} \int_{-\infty}^{\infty} g(p^{(t)}, \varrho; y)^k (1 - g(p^{(t)}, \varrho; y))^{m-k} dN(y)$$

for exchangeable Bernoulli variables (uniform portfolio) with

Credit risk in probability space – Bernoulli mixtures (2/2)

conditional PDs given by

$$g(p^{(t)}, \varrho; y) = \mathbb{P}[\mathbf{1}_{\sqrt{\varrho}Y + \sqrt{1-\varrho}\varepsilon_i < N^{-1}[p^{(t)}]} = 1 \mid Y = y]$$
$$= N\left[\frac{N^{-1}[p^{(t)}] - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right]$$

- $Y, \varepsilon_1, ..., \varepsilon_m \sim N(0, 1)$ i.i.d.
- $D_i^{(t)} = \{\sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \varepsilon_i < N^{-1}[p^{(t)}]\}$
- $\text{PD}_i^{(t)} = p_i^{(t)} = p^{(t)}$ (uniform PD / credit curve)
- ϱ uniform correlation of credit worthiness indices (CWI)

  $\text{CWI}_i = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \varepsilon_i$ (i=1,...,m)

Credit risk in probability space – loss distribution (1/3)

Portfolio loss:

w.r.t. time horizon t given by

$$L^{(t)} = \sum_{i=1}^{m} \mathrm{LGD}_i \times \mathrm{EAD}_i(\tau_i) \times \mathbf{1}_{\{\tau_i < t\}}$$

with additional credit risk parameters

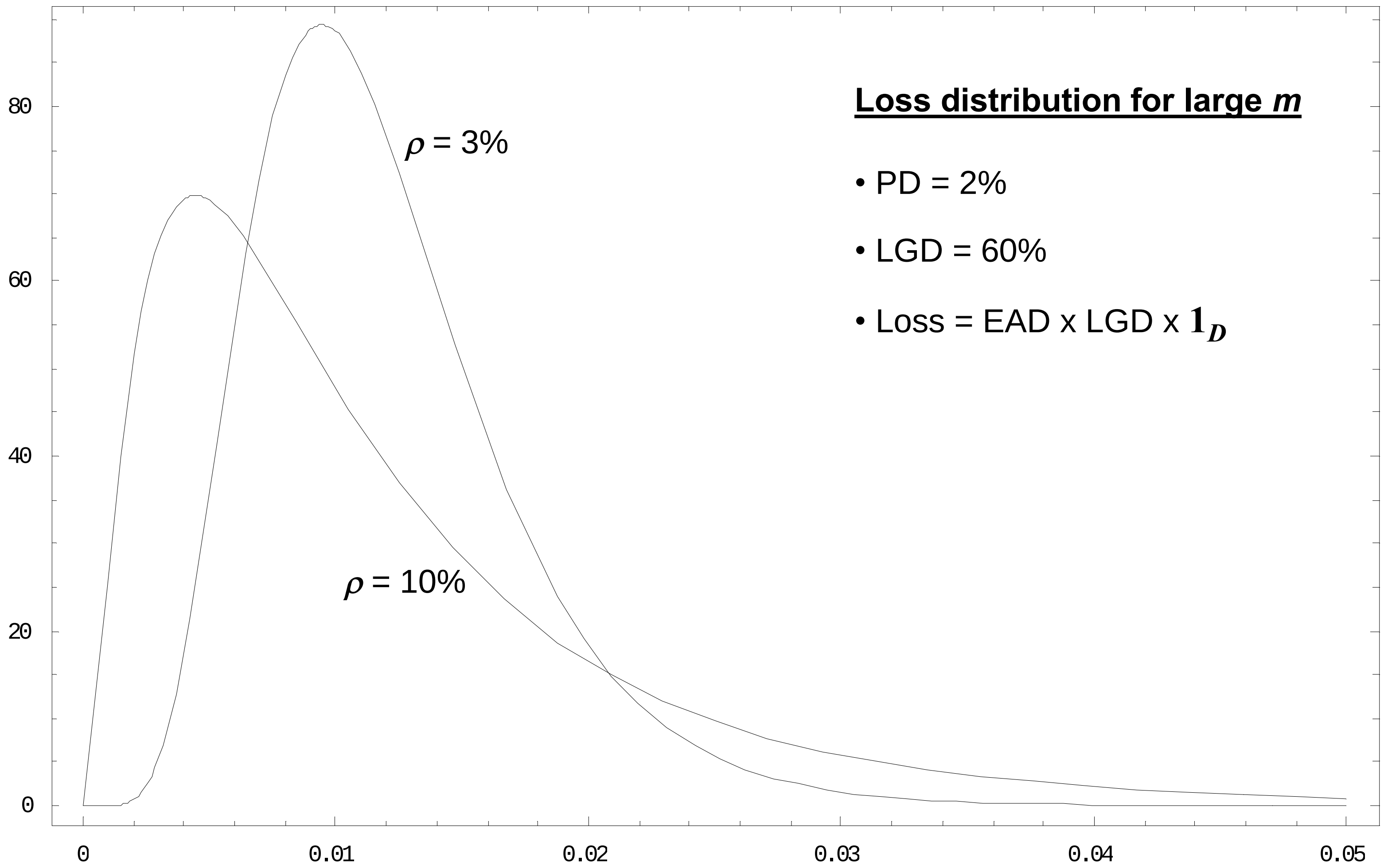
- LGD: **loss given default**

  expressing the severity of loss

- EAD: **exposure at default**

  quantifying the amount of money at risk at default time

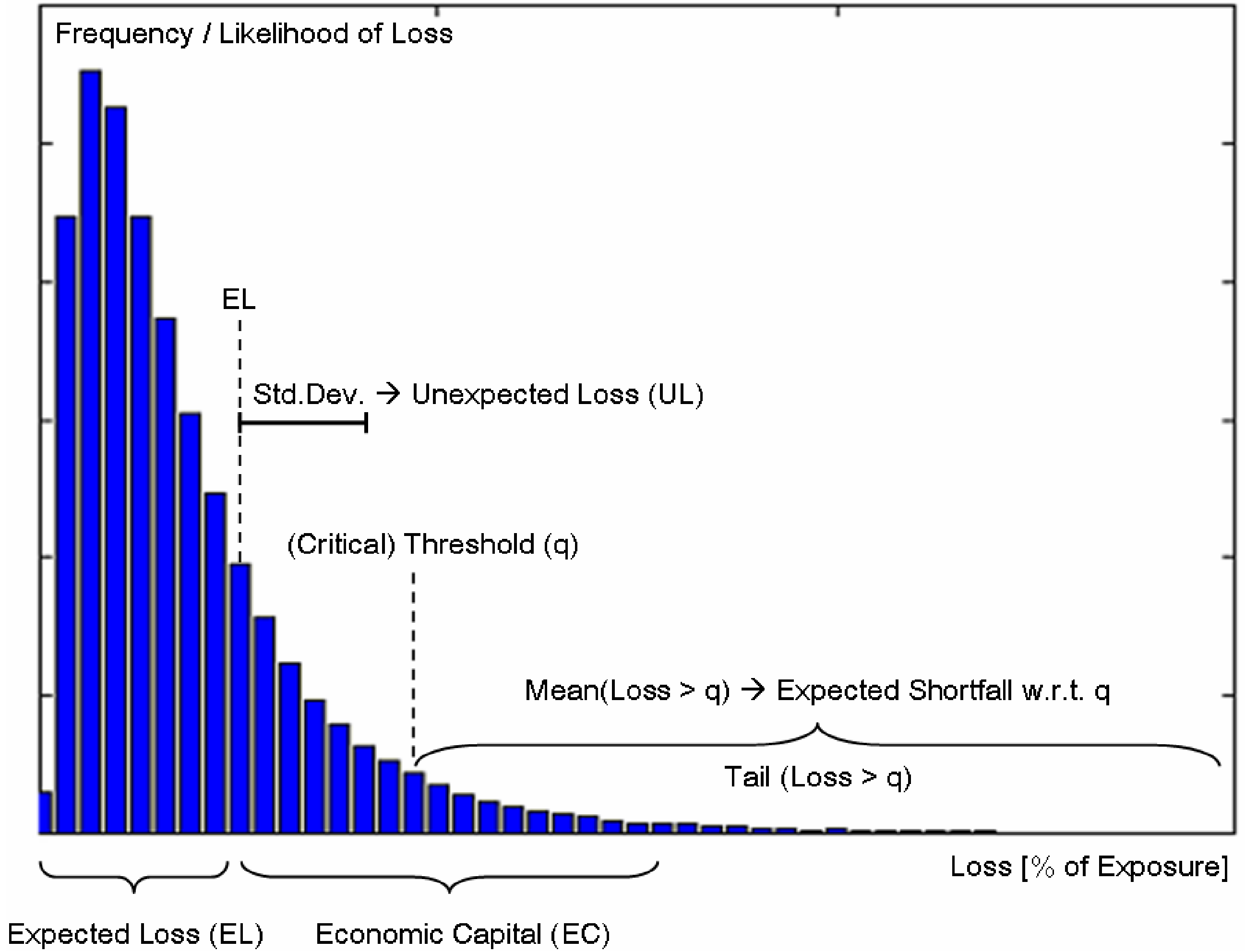
Credit risk in probability space – loss distribution (2/3)



Loss distribution for large *m*

- PD = 2%
- LGD = 60%
- Loss = EAD x LGD x $\mathbf{1}_D$

Credit risk in probability space – loss distribution (3/3)



Credit risk in probability space – default times (1/3)

Default times (standard definition):

marginal default time distributions

$$\mathbb{F}_i(t) = \mathbb{P}[\tau_i < t] = p_i^{(t)}$$

joint default times distribution

$$\mathbb{P}[\tau_1 < t_1, ..., \tau_m < t_m] = C(\mathbb{F}_1(t_1), ..., \mathbb{F}_m(t_m))$$

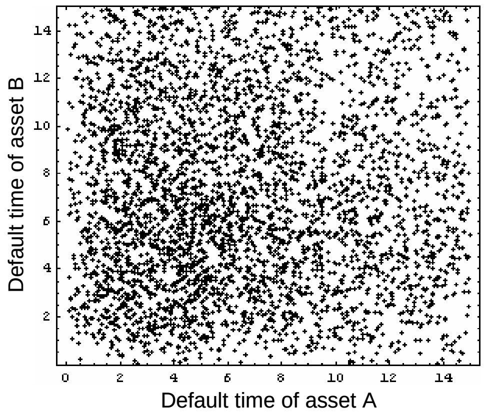
where $C : [0,1]^m \to [0,1]$ is the corresponding **copula** function

Typical examples in credit risk:

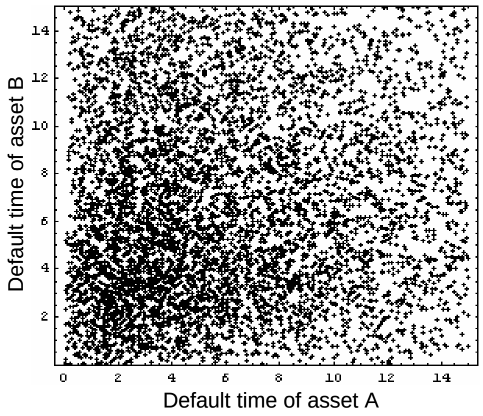
- Gaussian copula (standard Merton model)
- Student-t copula (fatter tails than Gauss)
- Archimedean copulas (Clayton, Gumbel, etc.)

Credit risk in probability space – default times (2/3)

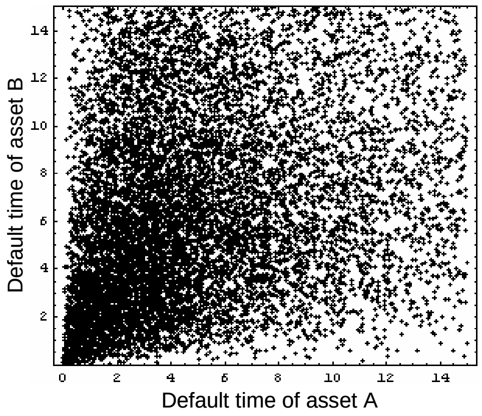
Gaussian copula, ρ = 0%



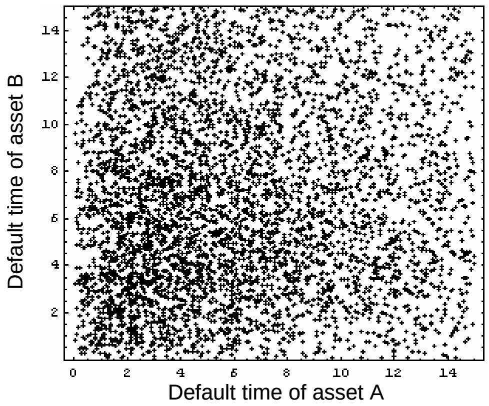
Gaussian copula, ρ = 30%



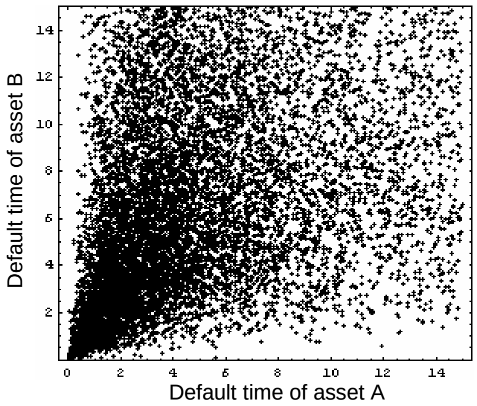
Gaussian copula, ρ = 70%



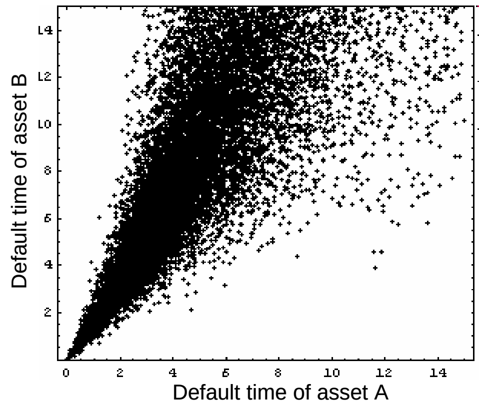
Clayton copula, η = 0.1



Clayton copula, η = 1

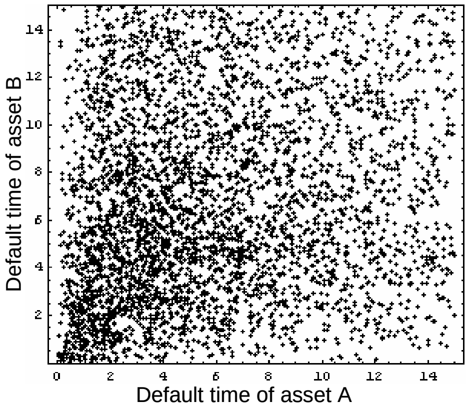


Clayton copula, η = 5

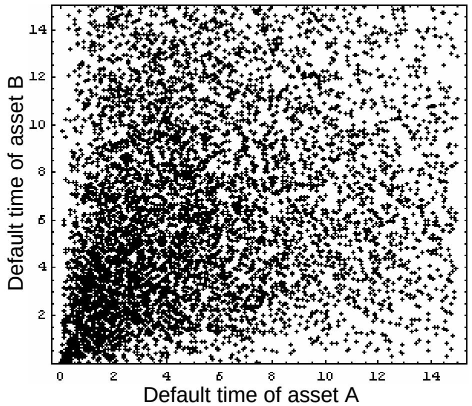


Credit risk in probability space – default times (3/3)

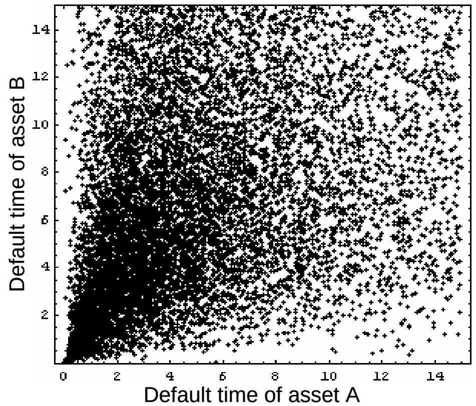
Student-t copula, ρ = 0%



Student-t copula, ρ = 30%



Student-t copula, ρ = 70%



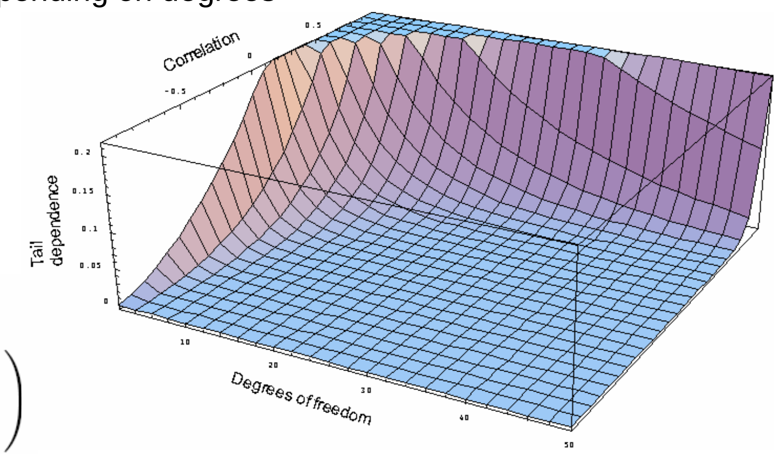
Student-t copula has upper and lower tail dependence depending on degrees of freedom and linear correlation

$$\lambda_U(X,Y) = \lim_{\gamma\to 1} \mathbb{P}[Y > \mathbb{F}_Y^{-1}(\gamma) \mid X > \mathbb{F}_X^{-1}(\gamma)]$$

$$\lambda_L(X,Y) = \lim_{\gamma\to 0} \mathbb{P}[Y \leq \mathbb{F}_Y^{-1}(\gamma) \mid X \leq \mathbb{F}_X^{-1}(\gamma)]$$

$$\lambda_U(X,Y) = \lambda_L(X,Y) =$$

$$= 2\left(1 - \Theta_{d+1}\left[\sqrt{d+1}\,\frac{\sqrt{1-\mathrm{Corr}[X,Y]}}{\sqrt{1+\mathrm{Corr}[X,Y]}}\right]\right)$$



Example: copula functions in the press ...

New angles

Logged in as: Christian Bluhm

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Deutsche trader dismissed

Hann Ho

Deutsche Bank dismissed credit trader Anshul Rustagi in January following an investigation into a £30 million overstatement sitting on his book. The German banking giant only picked up on the discrepancy in the third week of December while Rustagi was away on holiday – more than a month after the shortfall started accumulating.

Known as 'Rusty' to his colleagues and invariably described as a "young high flyer", Rustagi traded collateralised debt obligations (CDOs) for the German bank's credit derivatives unit in London, heavily using instruments such as credit default swap indexes. One broker in the iTraxx tranche interdealer market said Rustagi was "a top three trader in the Street in terms of volume".

The incident has sparked widescale speculation on the reasons for the discrepancy, with many questioning Deutsche Bank's internal risk management processes. There have also been some blogs on the internet accusing the bank of picking a scapegoat for its own failings. Both Deutsche Bank and Rustagi declined to comment on the case.

Others point to the very real danger of mis-marking a book. One head of trading at a US bank, who specialises in credit indexes, says the most likely sources for the discrepancy are either 'off-the-run' standardised iTraxx tranches (an older series of the iTraxx index) or non-standard iTraxx tranches (such as, say, the 1–4% tranche). Both are a lot less liquid than typical on-the-run indexes, which are widely quoted in the OTC markets.

"When you trade the standard tranches on an on-the-run index, such as a five-year 3–6% tranche, your marking is relatively easy. There is relatively transparent and liquid pricing in the broker market," explains the trader, who is based in London. However, the iTraxx indexes roll every six months – once they go off-the-run, their prices are not immediately available and their value is difficult to determine.

Non-standard tranches

"With non-standard tranches, things become more difficult because a trader needs to make some specific assumptions as they're not traded in the market. Obviously, there are some arbitrage relationships that need to hold, but even here there is some variability in the pricing and valuation," he continues. "Whether it was the fault of Rustagi only or a combination of people, we may never know."

Cyril Deretz, the London-based head of quantitative research at pricing and risk management specialist SunGard Reech, says that one problem relates to the way in which models are applied to non-standard CDOs. "Banks often use standard methodologies such as base correlation and the Gaussian copula model to price standard CDO tranches. The problem is that some people are also using them for non-standard tranches, such as bespoke portfolios," explains Deretz.

As a result of all the adverse publicity, there may be a negative impact on the structured credit market. "In the short term, it might scare a few clients off the market. But long term, I don't believe it will have any effect," says the trader. "All the banks' operations and risk departments are probably analysing their processes and asking, 'Could this have happened here?' The answer – even for those banks with very good processes – is yes, this can always happen. Their job, though, is to minimise the likelihood."

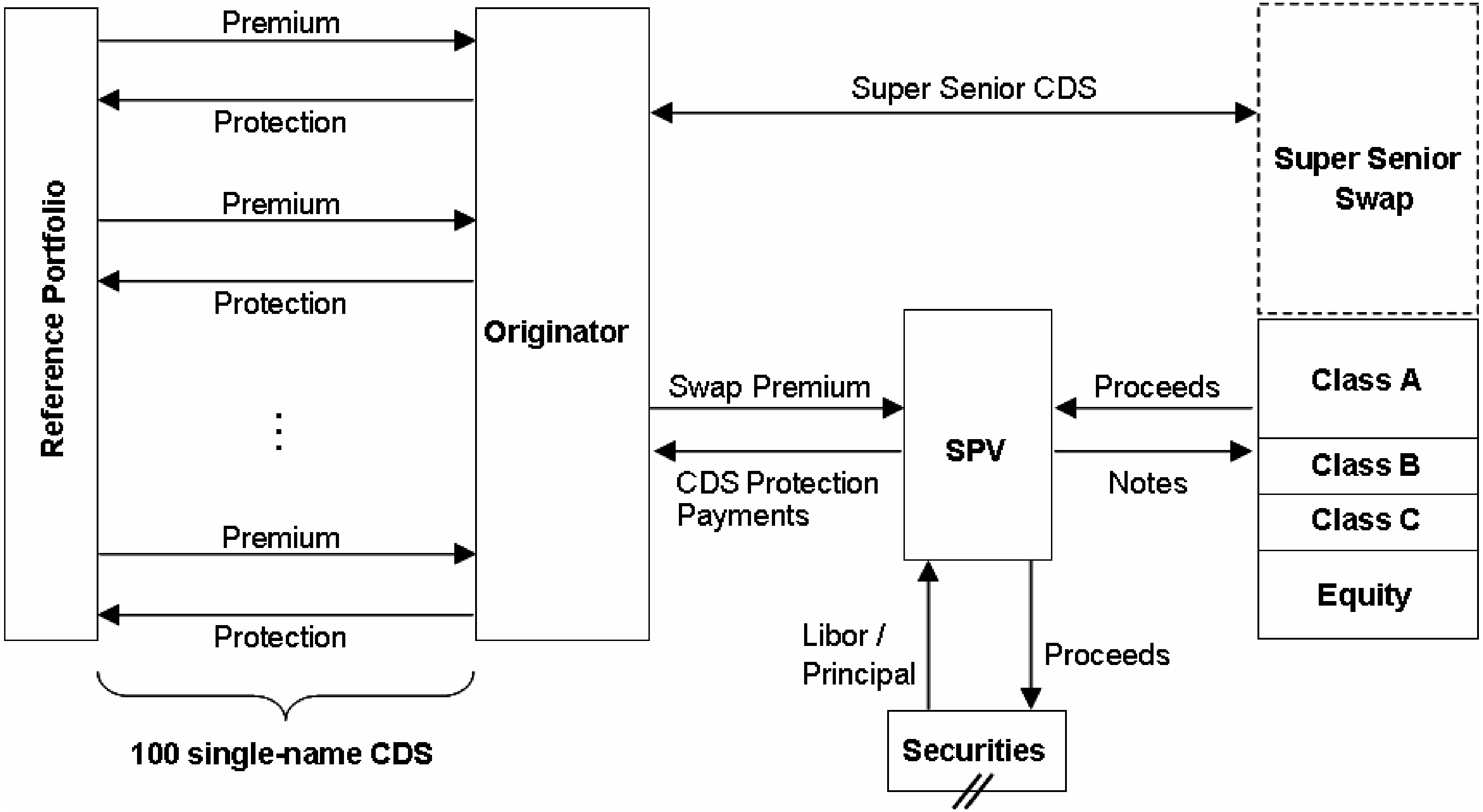
"Banks often use standard methodologies such as base correlation and the Gaussian copula model to price standard CDO tranches. The problem is that some people are also using them for non-standard tranches, such as bespoke portfolios" ...

Source: http://db.riskwaters.com/public/showPage.html?page=risk_story_NewAngles_53

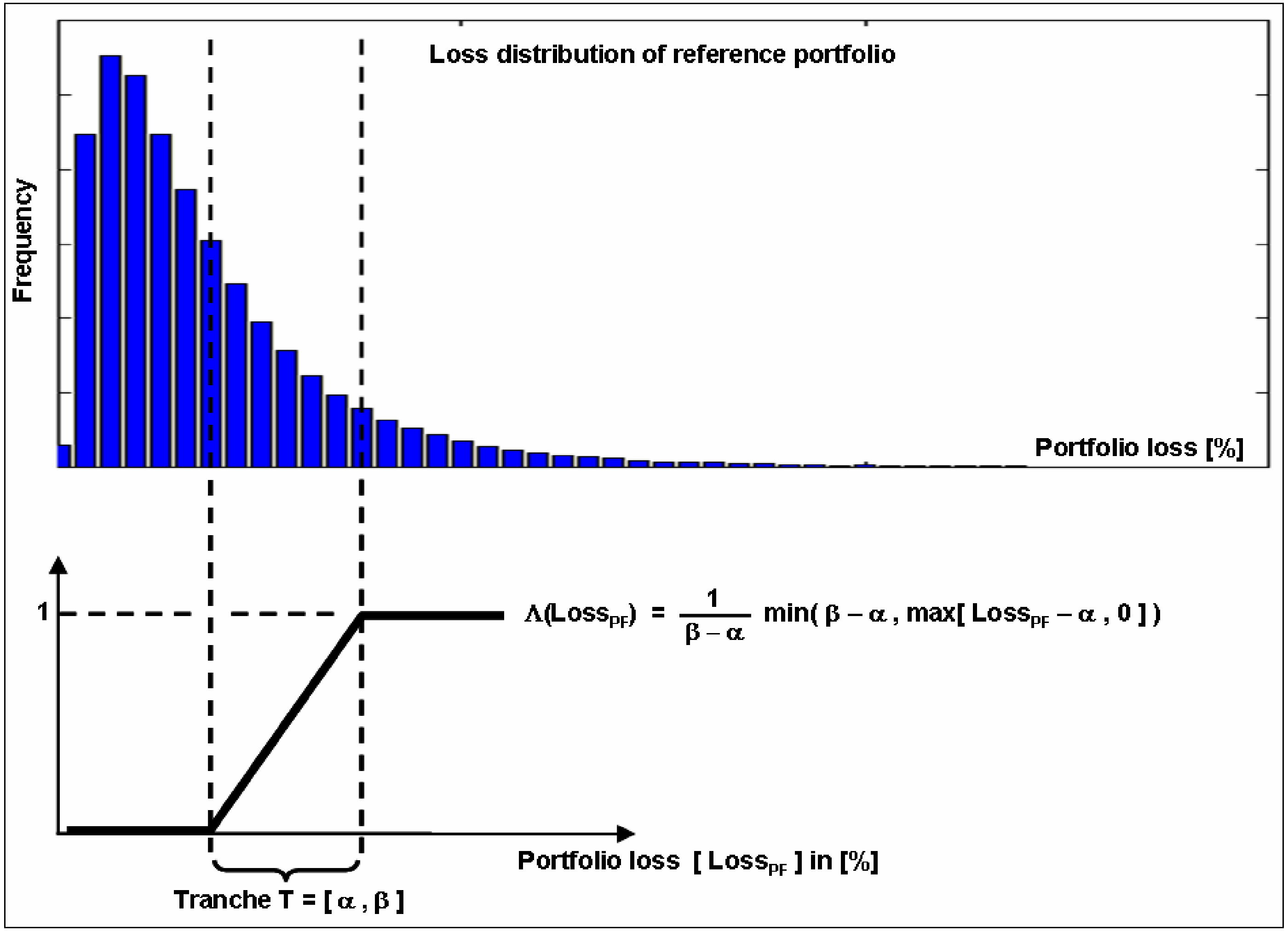
Next topic: correlation products and their motivations

Credit risk transfer	• Short position in the whole capital structure of a portfolio or in a certain part of the capital structure (e.g., a particular tranche) • Risk transfer instruments can be funded (e.g., CLN referenced to portfolio) or unfunded (swap-like; synthetic derivative transaction) • Reference names/portfolio range from highly liquid CDS (e.g., CDX, iTraxx) to SME or retail clients (e.g., CLOs, RMBS)
Spread & rating arbitrage	• Leverage of collected spreads on a pool of names via capital structure/tranching • Excess spread (e.g., left-over cash at end of cash flow waterfall) is distributed to equity/first losss piece investors • Credit enhancement (e.g., subordination, loss triggers) protect senior note holders
Funding benefits	• Funding spreads based on the risk of a pool of credit-risky instruments instead on rating of originating bank • Applies only to true-sale/funded transactions
Regulatory arbitrage	• Common motivation under Basel I regime • Will remain to be interesting in certain asset classes (e.g., high yield) • Difficult to capture under Basel II: correlation products in portfolio effect-free regime

Modeling structural elements – CDO tranches (1/4)



Modeling structural elements – CDO tranches (2/4)



Modeling structural elements – CDO tranches (3/4)

Consider a tranche $T_{\alpha,\beta} = [\alpha, \beta)$ with $0 \le \alpha < \beta \le 1$.

Tranche loss profile function:

$$L(T_{\alpha,\beta}) = \frac{1}{\beta - \alpha} \min[\max(L_{\text{PF}} - \alpha, 0), \beta - \alpha] \in [0, 1]$$

Expected loss in tranche:

$$\mathbb{E}[L(T_{\alpha,\beta})] = \frac{1}{\beta - \alpha} \int_0^1 \min[\max(x - \alpha, 0), \beta - \alpha] \mathrm{d}\mathbb{P}_{L_{\text{PF}}}(x)$$

Vasicek distribution example: $\mathbb{E}[L(T_{\alpha,\beta})] =$

$$= \frac{1}{\beta - \alpha} \int_0^1 \min\left[\max\left(N\left[\frac{N^{-1}[\text{PD}] - \sqrt{\varrho}\, y}{\sqrt{1 - \varrho}}\right] - \alpha, 0\right), \beta - \alpha\right] \mathrm{d}N(y)$$

PD uniform, ϱ uniform, for all names i, infinite granularity assumption

Modeling structural elements – CDO tranches (4/4)

Further aspects:

- purely synthetic versus cash flow transaction (waterfall & co.)
- default times approach versus multi-step approach
- clear conceptual separation between asset and liability side of transaction
- physical (risk assessment) versus risk-neutral (pricing) default probabilities
- role of rating agencies in CDO tranching and ratings → agency CDO models
- analytic or semi-analytic evaluation versus Monte Carlo simulation
- ...

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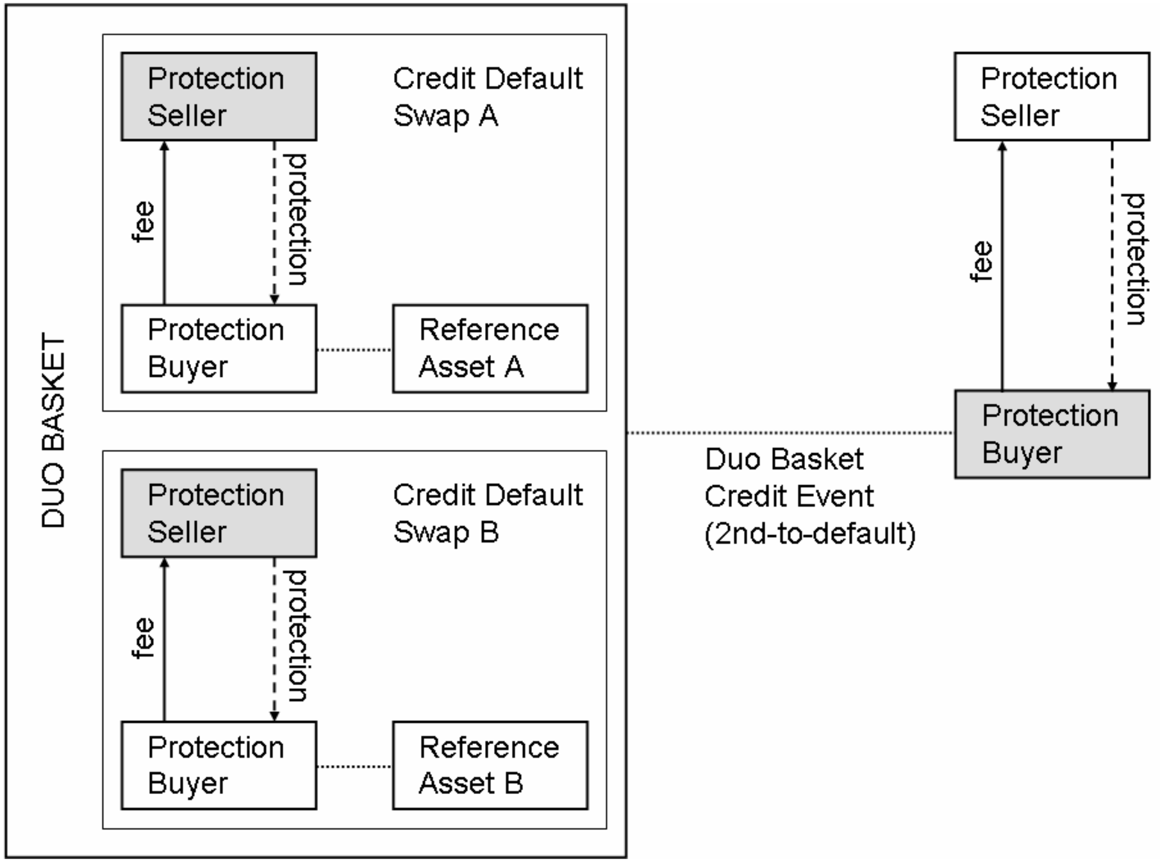
Presented examples

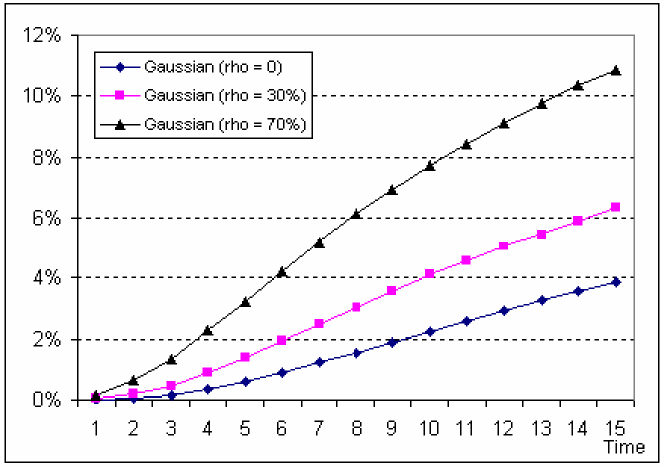
Example 1	2nd-to-default events in (duo) baskets
Example 2	Single-tranche CDOs and their implied correlations

Example 1: copula impact on basket 2nd-to-defaults (1/2)

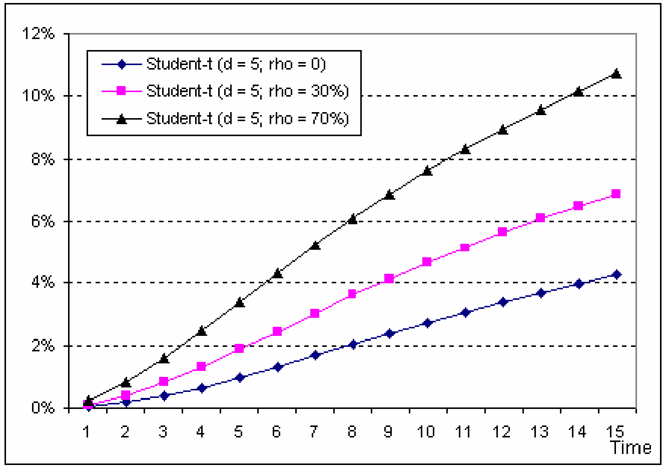


Example 1: copula impact on basket 2nd-to-defaults (2/2)

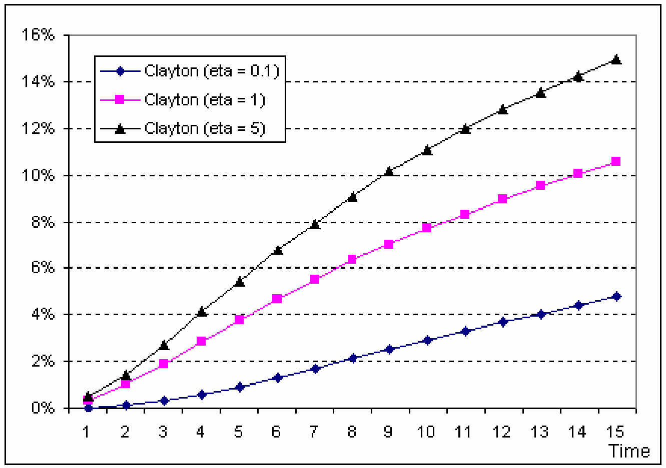
Second-to-default likelihoods for Gaussian copulas



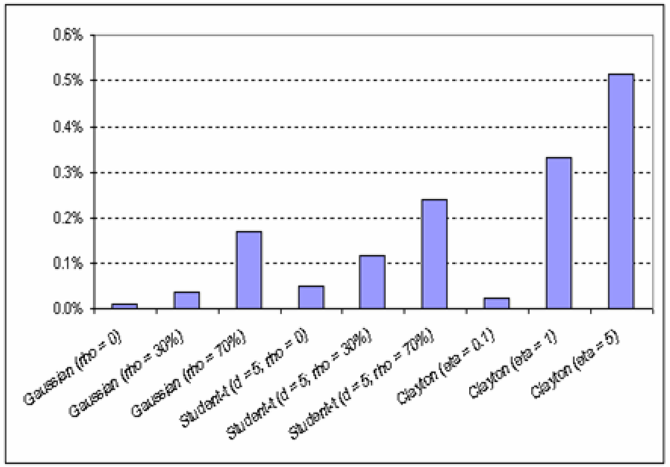
Second-to-default likelihoods for Student-t copulas



Second-to-default likelihoods for Clayton copulas



Second-to-default likelihoods (T = 1 year) for different copulas



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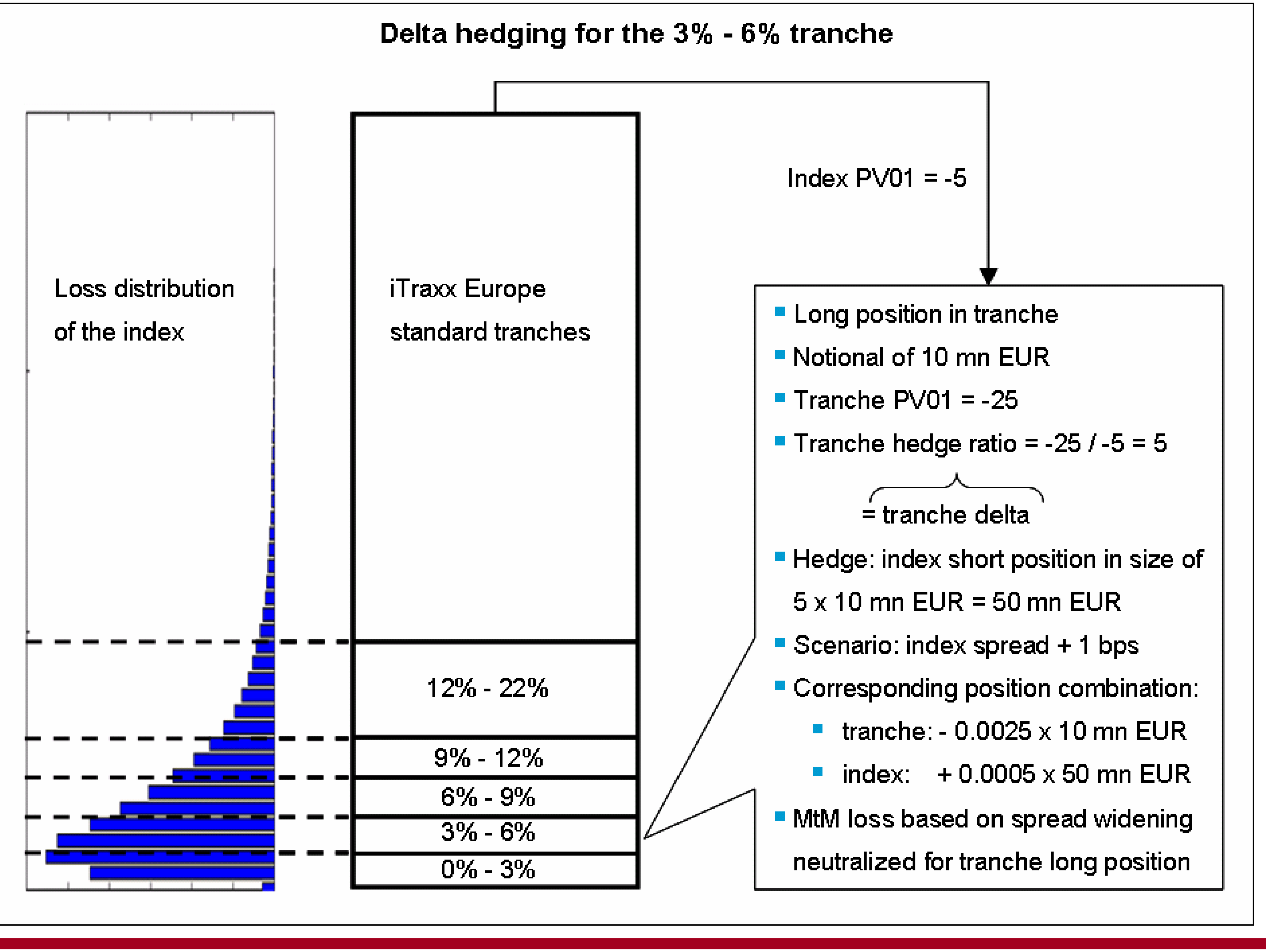
Example 2: CDS index STCDOs (1/7) - overview

- Single-tranche CDOs (STCDOs), sometimes also called „bespoken", are a recent and very interesting asset class attractive for investors looking for tailor-made credit risk positions
- STCDOs are negotiated basically in three steps:
  - Determination of a portfolio of underlying reference names, often from an „index" like iTraxx or CDX
  - Determination of an attachment point (lower strike) and detachment point (upper strike), hereby isolating part of the capital structure of the portfolio
  - Determination of a maturity of the transaction, most often 5 years
- STCDOs typically require delta hedging strategies to mitigate or eliminare mark-to-market losses based on spread movements (requires high liquidity of reference names)
- Besides spread fluctuations of the underlying reference names, STCDOs are sensitive to recovery scenarios and correlations/dependencies between names in the reference pool/index
- Delta hedging (spread neutrality) does not mean non-vulnerability w.r.t. defaults, correlation and recovery

Example 2: CDS index STCDOs (2/7) - delta hedging

STCDO example:

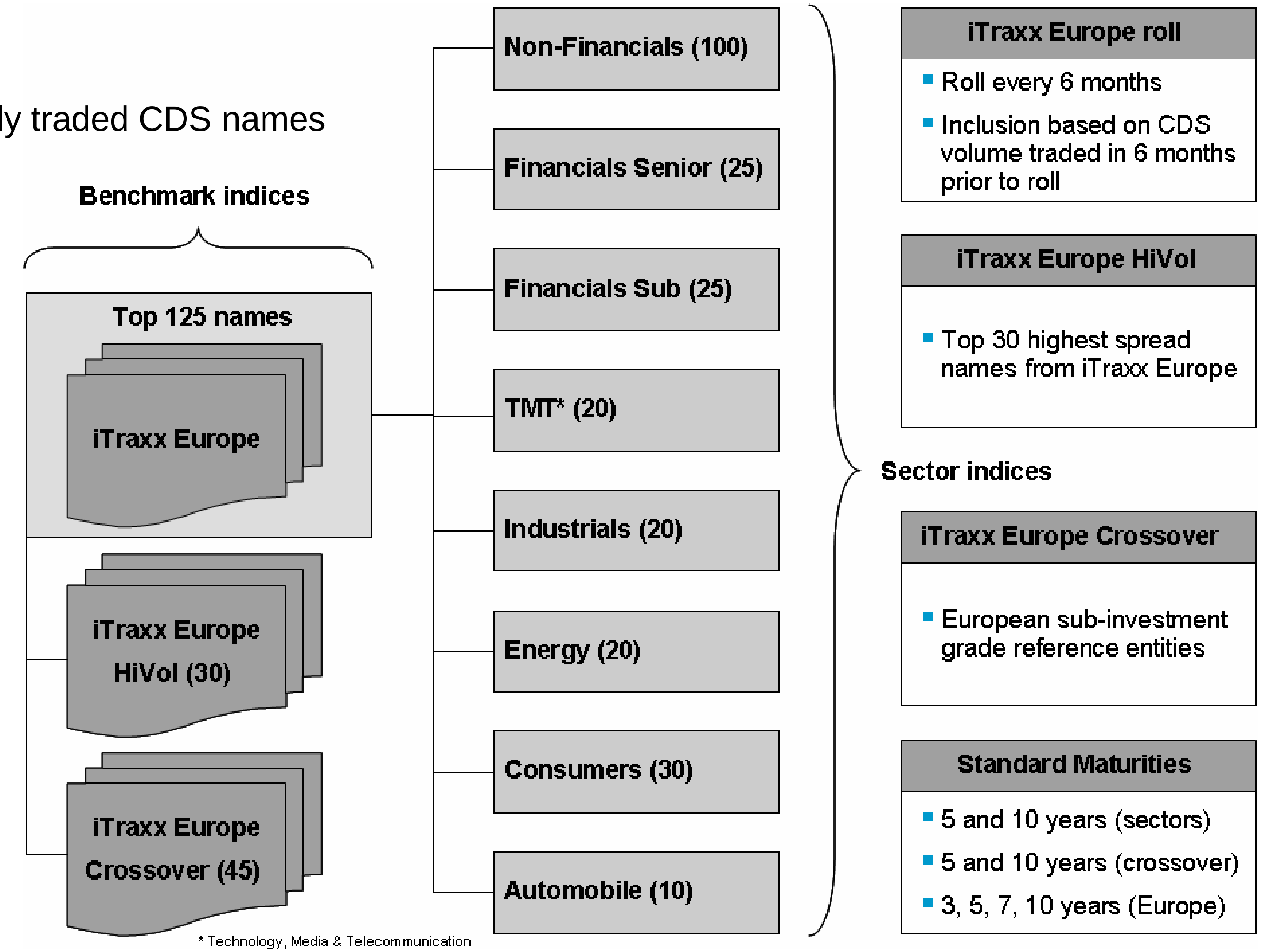
- Underlying names: iTraxx Europe Series 5
- Tranche: 3% - 6%
- Index spread: 32 bps
- Tranche spread: 65 bps
- Index PV01 = -5
- Tranche PV01 = -25
- Delta = 5
- Delta hedging can be expensive: here, we have a negative carry!



Example 2: CDS index STCDOs (3/7) - iTraxx

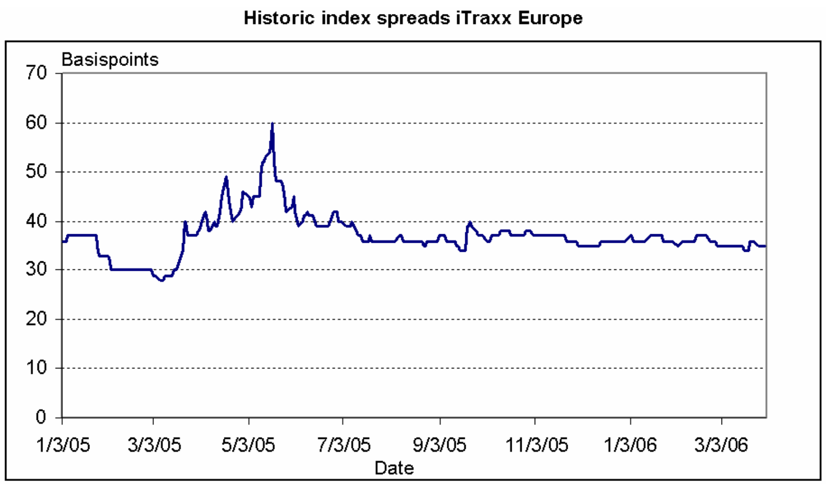
iTraxx example:

- 125 most liquid, most actively traded CDS names in Europe

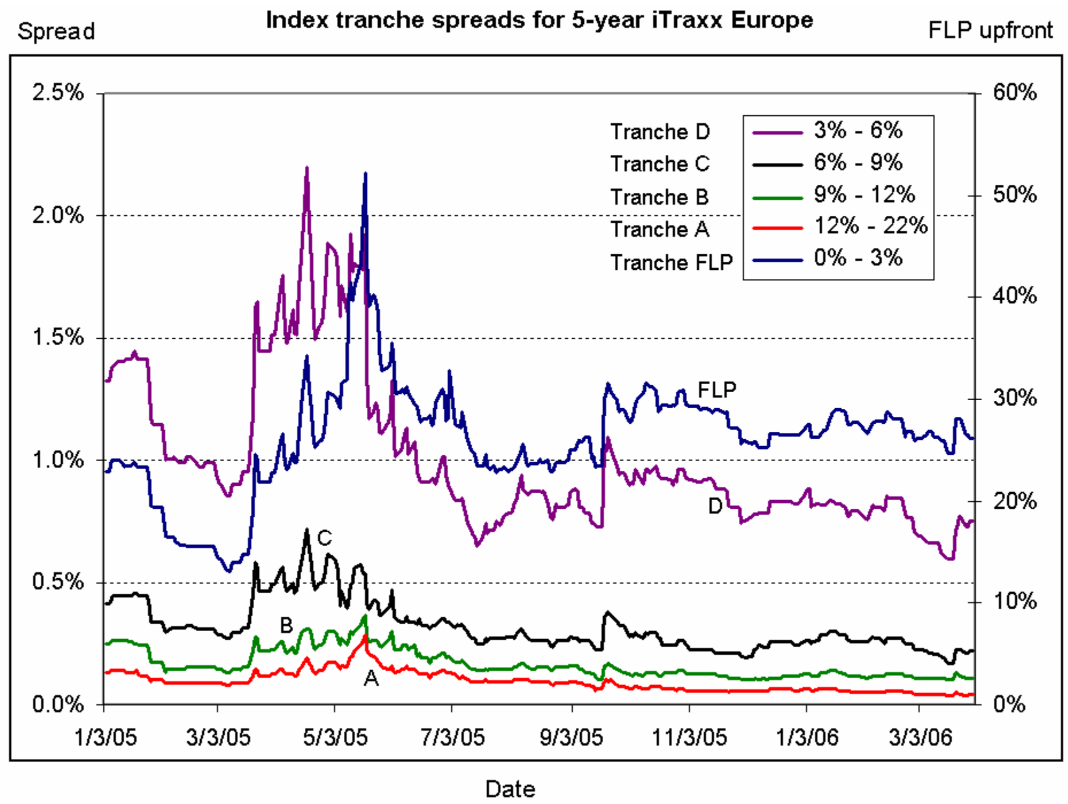


Example 2: CDS index STCDOs (4/7) - iTraxx prices

iTraxx index spread history



iTraxx index tranche spread history

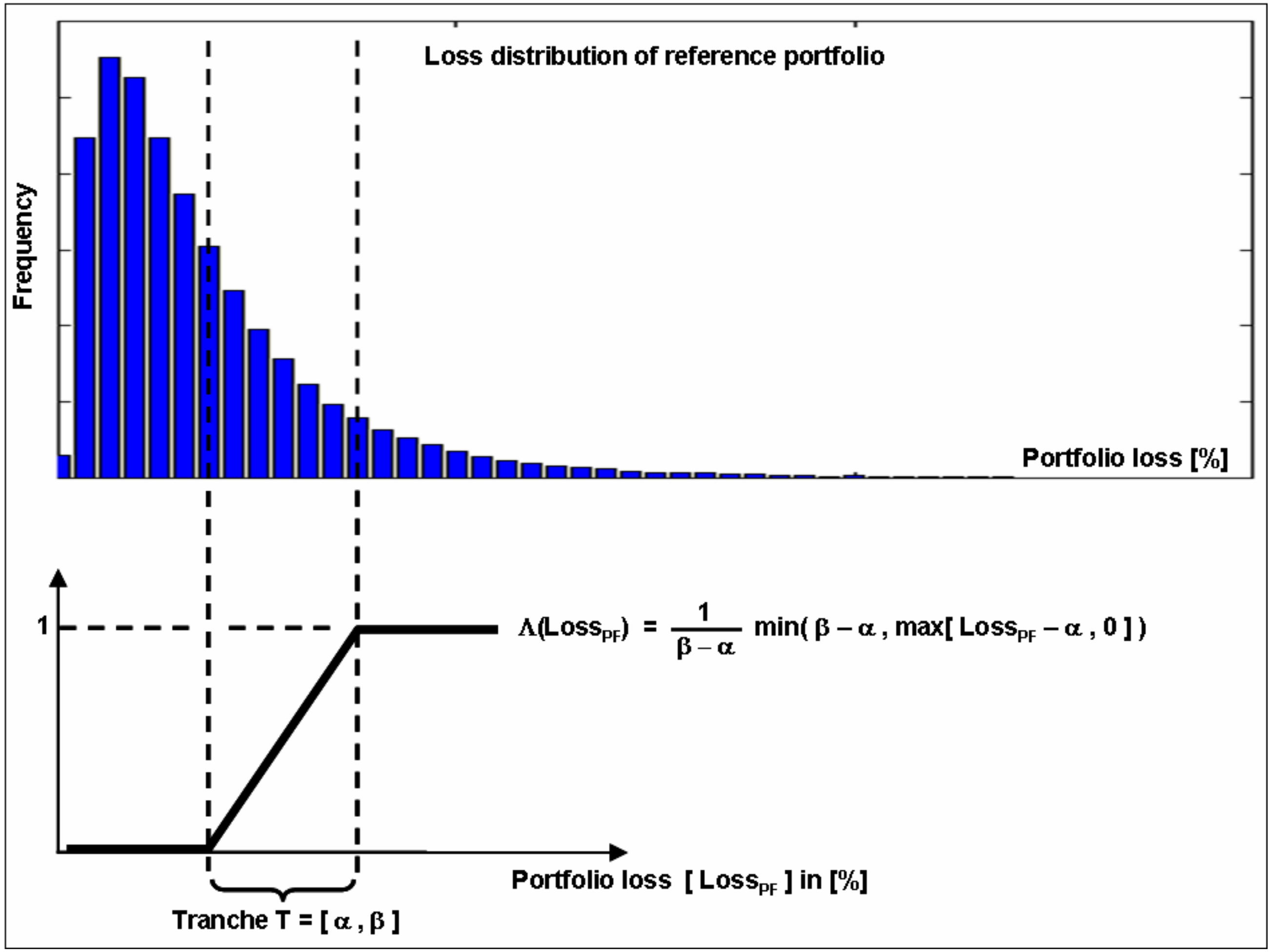


Example 2: CDS index STCDOs (5/7) - STCO std model

iTraxx example:

- Modeling standard: Gaussian copula model with plain vanilla tranche loss profile function
- Other models are considered for replication of market tranche spreads
- Implied correlation based on simplified CDO models

(see Slides 19 – 20)



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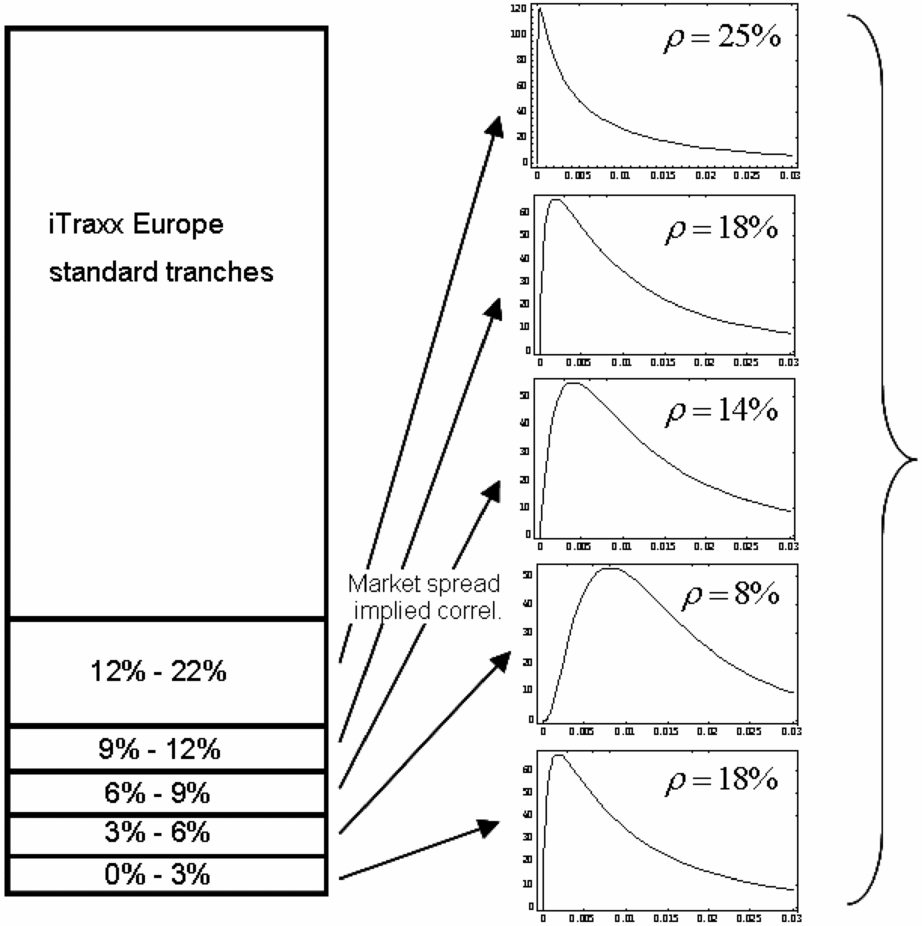
Example 2: CDS index STCDOs (6/7) - compound correl.

Compound (implied) correlation

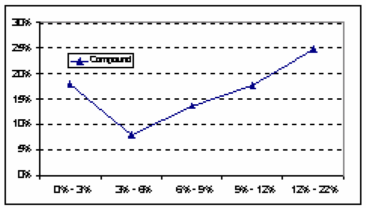


Compound correlation:

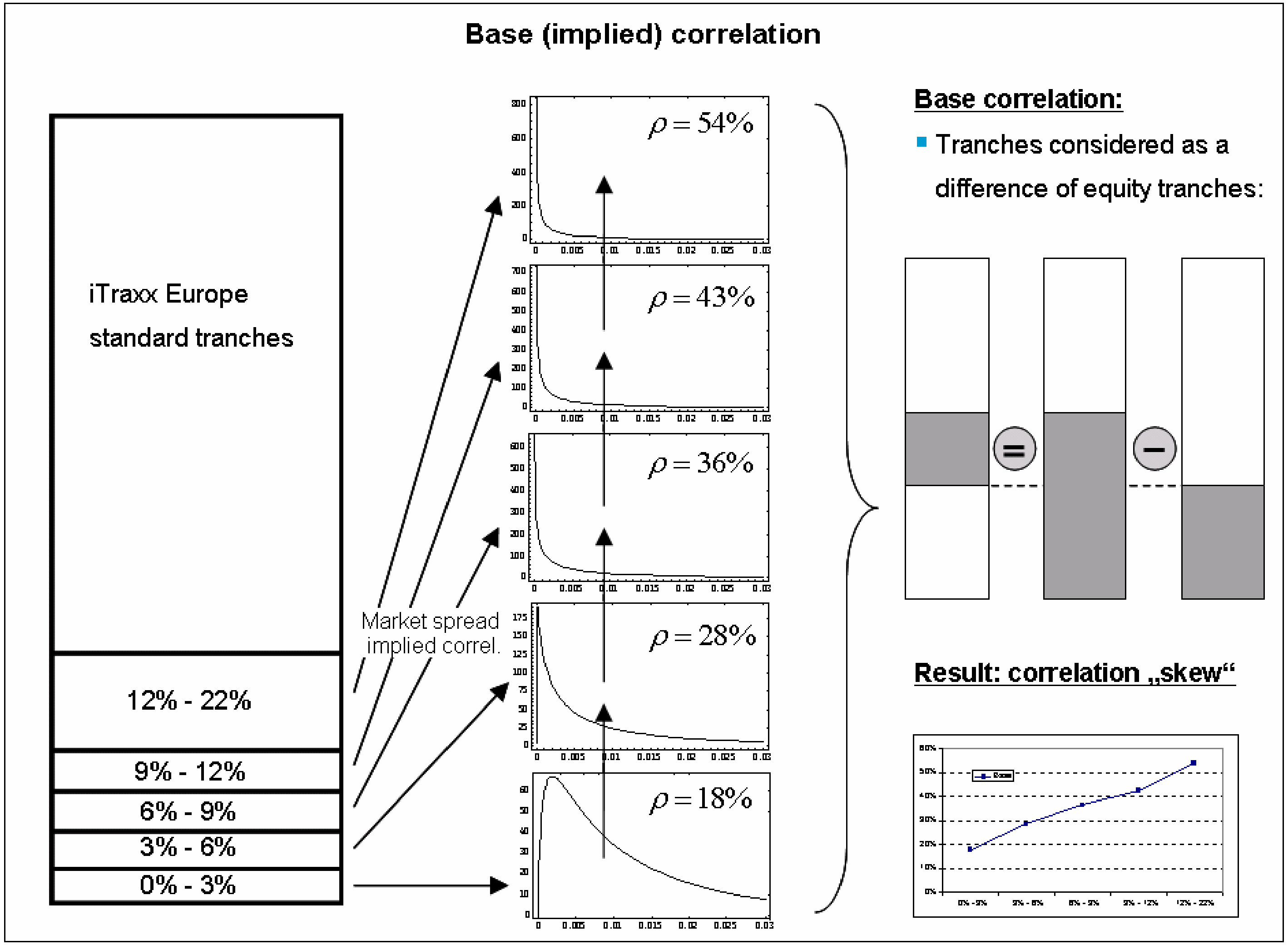
- Tranches considered in isolation of rest of the capital structure
- For every tranche, the market spread implied compound correlation determines a loss distr. of the reference pool
- Not a „one loss distr. fits all tranches“ approach

!

Result: correlation „smile“



Example 2: CDS index STCDOs (7/7): base correlation



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Wrap-up and conclusions

- Credit risk modeling based on probability concepts is the „bread & butter“ of Credit Portfolio Management (CPM)

- Over time, market standards in credit risk modeling have been developed

- However, sophisticated banks are typically one or several steps ahead in their internal credit risk modeling concepts → finer measurement enables finer steering & pricing

- Still, there are new and interesting things to discover in the credit risk modeling universe