A result of Pisier in the theory of operator spaces

Theorem (Pisier – 2011)

Let a, b positive real numbers such that $a > 0$. Let $z \in \mathbb{C}$. TFAE

(i) $\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, zk \rangle| \leq \sqrt{\langle h, ah \rangle} \cdot \sqrt{\langle k, bk \rangle}$ *for all* $h, k \in \mathbb{C}$.

(*Here,* $\langle u, v \rangle := u\bar{v}$ *for all* $u, v \in \mathbb{C}$.)

A result of Pisier in the theory of operator spaces

Theorem (Pisier – 2011)

Let a, b *positive real numbers such that* $a > 0$. *Let* $z \in \mathbb{C}$. *TFAE*

(i) $\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, zk \rangle| \leq \sqrt{\langle h, ah \rangle} \cdot \sqrt{\langle k, bk \rangle}$ *for all* $h, k \in \mathbb{C}$.

(*Here,* $\langle u, v \rangle := u\bar{v}$ *for all* $u, v \in \mathbb{C}$.)

Corollary

Let $z \in \mathbb{C}$. *TFAE*

(i) $\begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$ *is positive semidefinite.*

(ii) $|z| \leq 1$.

Pisier's proof revisited (I)

Proof of Pisier's Theorem.

Firstly, we prove "(i) $\Rightarrow$ (ii)". Clearly condition (i) is equivalent to the following inequality:

$$\langle h, ah\rangle + 2\Re\big(\langle h, zk\rangle\big) + \langle k, bk\rangle \geq 0$$

for all $h, k \in \mathbb{C}$.

Pisier's proof revisited (I)

Proof of Pisier's Theorem.

Firstly, we prove "(i) $\Rightarrow$ (ii)". Clearly condition (i) is equivalent to the following inequality:

$$\langle h, ah\rangle + 2\Re\big(\langle h, zk\rangle\big) + \langle k, bk\rangle \geq 0$$

for all $h, k \in \mathbb{C}$. Since this equality is true for all $h \in \mathbb{C}$, we equivalently obtain

$$|\lambda|^2\langle h, ah\rangle + 2\Re\big(\lambda\langle h, zk\rangle\big) + \langle k, bk\rangle \geq 0$$

for all $h, k, \lambda \in \mathbb{C}$.

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h,zk \rangle}}{\langle h,ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h, zk \rangle}}{\langle h, ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

We now are going to prove "(ii) $\Rightarrow$ (i)" (yet without a direct application of the arithmetic-geometric mean inequality).

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h,zk \rangle}}{\langle h,ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

We now are going to prove "(ii) $\Rightarrow$ (i)" (yet without a direct application of the arithmetic-geometric mean inequality).

To this end, we continue with a lemma - which is of its own interest - and easy to prove. It namely generalises the arithmetic-geometric mean inequality.

Pisier's proof revisited (III)

Lemma

Let α and β be positive real numbers such that $\alpha > 0$. Consider the function $f_{\alpha,\beta} : (0, \infty) \longrightarrow [0, \infty)$, defined through

$$f_{\alpha,\beta}(s) := s^2\alpha + \frac{1}{s^2}\beta \qquad (s > 0)\,.$$

Then $f_{\alpha,\beta}$ is a smooth function which attains its global minimum at $s^* := \sqrt[4]{\frac{\beta}{\alpha}}$, given by

$$f_{\alpha,\beta}(s^*) = 2\sqrt{\alpha}\sqrt{\beta}\,.$$

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \stackrel{(!)}{=} \alpha + \beta$.

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \overset{(!)}{=} \alpha + \beta$*.*

Proof of Pisier's Theorem ctd.

Let $h, k \in \mathbb{C}$ arbitrary. Put $\alpha := \langle h, ah\rangle$ and $\beta := \langle k, bk\rangle$.

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \overset{(!)}{=} \alpha + \beta$*.*

Proof of Pisier's Theorem ctd.

Let $h, k \in \mathbb{C}$ arbitrary. Put $\alpha := \langle h, ah\rangle$ and $\beta := \langle k, bk\rangle$. Due to the assumed property (ii), the previous lemma therefore implies that

$$\begin{aligned} |\langle h, zk\rangle| &\leq \sqrt{\langle h, ah\rangle} \cdot \sqrt{\langle k, bk\rangle} \\ &= \frac{1}{2} f_{\alpha,\beta}(s^*) \\ &\leq \frac{1}{2} f_{\alpha,\beta}(1) = \frac{1}{2}\big(\langle h, ah\rangle + \langle k, bk\rangle\big). \end{aligned}$$

Pisier's proof revisited (V)

Proof of Pisier's Theorem ctd.

Hence, for all $h, k \in \mathbb{C}$ we have

$$\left|\Re\big(\langle h, zk\rangle\big)\right| \leq |\langle h, zk\rangle| \leq \frac{1}{2}\big(\langle h, ah\rangle + \langle k, bk\rangle\big),$$

implying that in particular

$$-\big(\langle h, ah\rangle + \langle k, bk\rangle\big) \leq 2\Re\big(\langle h, zk\rangle\big)$$

which clearly is equivalent to condition (i). $\square$