A result of Pisier in the theory of operator spaces

Theorem (Pisier – 2011)

Let a, b be positive real numbers. Let $z \in \mathbb{C}$. TFAE

(i) $\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, zk\rangle| \leq \sqrt{\langle h, ah\rangle} \cdot \sqrt{\langle k, bk\rangle}$ *for all* $h, k \in \mathbb{C}$.

(*Here,* $\langle u, v\rangle := u\bar{v}$ *for all* $u, v \in \mathbb{C}$.)

A result of Pisier in the theory of operator spaces

Theorem (Pisier – 2011)

Let a, b be positive real numbers. Let $z \in \mathbb{C}$. TFAE

(i) $\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, zk\rangle| \leq \sqrt{\langle h, ah\rangle} \cdot \sqrt{\langle k, bk\rangle}$ *for all* $h, k \in \mathbb{C}$.

(*Here,* $\langle u, v\rangle := u\bar{v}$ *for all* $u, v \in \mathbb{C}$.)

Corollary

Let $z \in \mathbb{C}$. TFAE

(i) $\begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$ *is positive semidefinite.*

(ii) $|z| \leq 1$.

Pisier's proof revisited (I)

Proof of Pisier's Theorem.

Firstly, we prove "(i) $\Rightarrow$ (ii)". To this end, let us assume WLOG that $a > 0$ (else consider $0 < a_n := a + \frac{1}{n}$ and take the limit on both sides of the resulting inequality (ii) (applied to a_n) as $n \to \infty$).

Pisier's proof revisited (I)

Proof of Pisier's Theorem.

Firstly, we prove "(i) $\Rightarrow$ (ii)". To this end, let us assume WLOG that $a > 0$ (else consider $0 < a_n := a + \frac{1}{n}$ and take the limit on both sides of the resulting inequality (ii) (applied to a_n) as $n \to \infty$).

Clearly condition (i) is equivalent to the following inequality:

$$\langle h, ah \rangle + 2\mathfrak{R}\big(\langle h, zk \rangle\big) + \langle k, bk \rangle \geq 0$$

for all $h, k \in \mathbb{C}$.

Proof of Pisier's Theorem.

Firstly, we prove "(i) $\Rightarrow$ (ii)". To this end, let us assume WLOG that $a > 0$ (else consider $0 < a_n := a + \frac{1}{n}$ and take the limit on both sides of the resulting inequality (ii) (applied to a_n) as $n \to \infty$).

Clearly condition (i) is equivalent to the following inequality:

$$\langle h, ah\rangle + 2\Re\big(\langle h, zk\rangle\big) + \langle k, bk\rangle \geq 0$$

for all $h, k \in \mathbb{C}$. Since this equality is true for all $h \in \mathbb{C}$, we equivalently obtain

$$|\lambda|^2\langle h, ah\rangle + 2\Re\big(\lambda\langle h, zk\rangle\big) + \langle k, bk\rangle \geq 0$$

for all $h, k, \lambda \in \mathbb{C}$.

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h,zk \rangle}}{\langle h,ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h,zk \rangle}}{\langle h,ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

We now are going to prove "(ii) $\Rightarrow$ (i)" (yet without a direct application of the arithmetic-geometric mean inequality).

Pisier's proof revisited (II)

Proof of Pisier's Theorem ctd.

Clearly, (ii) follows if $h = 0$. So, let us assume that $h \neq 0$. Since by assumption $a > 0$, we also have $\langle h, ah \rangle = |h|^2 a > 0$. Hence, $\lambda := -\frac{\overline{\langle h,zk \rangle}}{\langle h,ah \rangle}$ is well-defined. A remaining (yet elementary) calculation directly leads to statement (ii).

We now are going to prove "(ii) $\Rightarrow$ (i)" (yet without a direct application of the arithmetic-geometric mean inequality).

To this end, we continue with a lemma - which is of its own interest - and easy to prove. It namely generalises the arithmetic-geometric mean inequality.

Pisier's proof revisited (III)

Lemma

Let α and β be positive real numbers such that $\alpha > 0$. Consider the function $f_{\alpha,\beta} : (0,\infty) \longrightarrow [0,\infty)$, defined through

$$f_{\alpha,\beta}(s) := s^2\alpha + \frac{1}{s^2}\beta \qquad (s > 0)\,.$$

Then $f_{\alpha,\beta}$ is a smooth function which attains its global minimum at $s^* := \sqrt[4]{\frac{\beta}{\alpha}}$, given by

$$f_{\alpha,\beta}(s^*) = 2\sqrt{\alpha}\sqrt{\beta}\,.$$

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \overset{(!)}{=} \alpha + \beta = f_{\alpha,\beta}(1)$.

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \overset{(!)}{=} \alpha + \beta = f_{\alpha,\beta}(1)$.

Proof of Pisier's Theorem ctd.

Let $h, k \in \mathbb{C}$ arbitrary. Put $\alpha := \langle h, ah \rangle$ and $\beta := \langle k, bk \rangle$.

Pisier's proof revisited (IV)

Remark

Notice that $f_{\alpha,\beta}((s^*)^2) \overset{(!)}{=} \alpha + \beta = f_{\alpha,\beta}(1)$.

Proof of Pisier's Theorem ctd.

Let $h, k \in \mathbb{C}$ arbitrary. Put $\alpha := \langle h, ah \rangle$ and $\beta := \langle k, bk \rangle$. Due to the assumed property (ii), the previous lemma therefore implies that

$$\begin{aligned} |\langle h, zk \rangle| &\leq \sqrt{\langle h, ah \rangle} \cdot \sqrt{\langle k, bk \rangle} \\ &= \frac{1}{2} f_{\alpha,\beta}(s^*) \\ &\leq \frac{1}{2} f_{\alpha,\beta}(1) = \frac{1}{2}\big(\langle h, ah \rangle + \langle k, bk \rangle\big). \end{aligned}$$

Pisier's proof revisited (V)

Proof of Pisier's Theorem ctd.

Hence, for all $h, k \in \mathbb{C}$ we have

$$\left|\Re\big(\langle h, zk\rangle\big)\right| \leq |\langle h, zk\rangle| \leq \frac{1}{2}\big(\langle h, ah\rangle + \langle k, bk\rangle\big),$$

implying that in particular

$$-\big(\langle h, ah\rangle + \langle k, bk\rangle\big) \leq 2\Re\big(\langle h, zk\rangle\big)$$

which clearly is equivalent to condition (i). $\square$

Pisier's proof revisited (V)

Proof of Pisier's Theorem ctd.

Hence, for all $h, k \in \mathbb{C}$ we have

$$\left|\Re\big(\langle h, zk\rangle\big)\right| \le |\langle h, zk\rangle| \le \frac{1}{2}\big(\langle h, ah\rangle + \langle k, bk\rangle\big),$$

implying that in particular

$$-\big(\langle h, ah\rangle + \langle k, bk\rangle\big) \le 2\Re\big(\langle h, zk\rangle\big)$$

which clearly is equivalent to condition (i). $\square$

Remark

Notice that we don't have to assume that $b \neq 0$.

Pisier's proof revisited (V)

Proof of Pisier's Theorem ctd.

Hence, for all $h, k \in \mathbb{C}$ we have

$$\left|\Re\big(\langle h, zk\rangle\big)\right| \leq |\langle h, zk\rangle| \leq \frac{1}{2}\big(\langle h, ah\rangle + \langle k, bk\rangle\big)\,,$$

implying that in particular

$$-\big(\langle h, ah\rangle + \langle k, bk\rangle\big) \leq 2\Re\big(\langle h, zk\rangle\big)$$

which clearly is equivalent to condition (i). $\square$

Remark

Notice that we don't have to assume that $b \neq 0$.

By mimicking the structure of this proof, we similarly obtain a further important result:

On positive semidefinite block matrices

Theorem

Let $n \in \mathbb{N}$ *and* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $A, B \in \mathbb{M}(n \times n; \mathbb{F})$ *be positive semidefinite matrices. Let* $Z \in \mathbb{M}(n \times n; \mathbb{F})$. *TFAE*

(i) $\begin{pmatrix} A & Z \\ Z^* & B \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, Zk \rangle| \leq \sqrt{\langle h, Ah \rangle} \cdot \sqrt{\langle k, Bk \rangle} = \|A^{1/2}h\| \cdot \|B^{1/2}k\|$ *for all* $h, k \in \mathbb{F}^n$.

On positive semidefinite block matrices

Theorem

Let $n \in \mathbb{N}$ *and* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $A, B \in \mathbb{M}(n \times n; \mathbb{F})$ *be positive semidefinite matrices. Let* $Z \in \mathbb{M}(n \times n; \mathbb{F})$. *TFAE*

(i) $\begin{pmatrix} A & Z \\ Z^* & B \end{pmatrix}$ *is positive semidefinite.*

(ii) $|\langle h, Zk \rangle| \leq \sqrt{\langle h, Ah \rangle} \cdot \sqrt{\langle k, Bk \rangle} = \|A^{1/2}h\| \cdot \|B^{1/2}k\|$ *for all* $h, k \in \mathbb{F}^n$.

Corollary

Let $\alpha \geq 0$ *and* $Z \in \mathbb{M}(n \times n; \mathbb{R})$. *TFAE*

(i) $\begin{pmatrix} \alpha E_n & Z \\ Z^* & \alpha E_n \end{pmatrix}$ *is positive semidefinite.*

(ii) $\|Z\| \leq \alpha$.

Here, $\|Z\|$ *denotes the standard operator norm of* Z.

Norm of the Schur multiplier of a positive semidefinite matrix I

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Next, we are going to work with the set of all those $n \times n$-matrices which canonically arise from the normed vector space $\mathcal{B}\left(l_2^n\right)$ of all bounded linear operators on $l_2^n := (\mathbb{F}^n, \|\cdot\|_2)$.

Norm of the Schur multiplier of a positive semidefinite matrix I

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Next, we are going to work with the set of all those $n \times n$-matrices which canonically arise from the normed vector space $\mathcal{B}\left(l_2^n\right)$ of all bounded linear operators on $l_2^n := (\mathbb{F}^n, \|\cdot\|_2)$.

Let $A \in \mathbb{M}(n \times n; \mathbb{F})$. Consider the Schur multiplier $S_A : \mathcal{B}\big(l_2^n\big) \longrightarrow \mathcal{B}\big(l_2^n\big)$, defined as

$$S_A(B) := A * B \qquad (B \in \mathcal{B}\big(l_2^n\big)),$$

where $(A * B)_{ij} := A_{ij} B_{ij}$ for all $i, j \in [n]$ (Schur multiplication).

Norm of the Schur multiplier of a positive semidefinite matrix I

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Next, we are going to work with the set of all those $n \times n$-matrices which canonically arise from the normed vector space $\mathcal{B}\left(l_2^n\right)$ of all bounded linear operators on $l_2^n := (\mathbb{F}^n, \|\cdot\|_2)$.

Let $A \in \mathbb{M}(n \times n; \mathbb{F})$. Consider the Schur multiplier $S_A : \mathcal{B}\big(l_2^n\big) \longrightarrow \mathcal{B}\big(l_2^n\big)$, defined as

$$S_A(B) := A * B \qquad (B \in \mathcal{B}\big(l_2^n\big)),$$

where $(A * B)_{ij} := A_{ij} B_{ij}$ for all $i, j \in [n]$ (Schur multiplication).

Proposition

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $A \in \mathbb{M}(n \times n; \mathbb{F})$ be positive semidefinite.

Norm of the Schur multiplier of a positive semidefinite matrix I

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Next, we are going to work with the set of all those $n \times n$-matrices which canonically arise from the normed vector space $\mathcal{B}\left(l_2^n\right)$ of all bounded linear operators on $l_2^n := (\mathbb{F}^n, \|\cdot\|_2)$.

Let $A \in \mathbb{M}(n \times n; \mathbb{F})$. Consider the Schur multiplier $S_A : \mathcal{B}(l_2^n) \longrightarrow \mathcal{B}(l_2^n)$, defined as

$$S_A(B) := A * B \qquad (B \in \mathcal{B}(l_2^n)),$$

where $(A * B)_{ij} := A_{ij} B_{ij}$ for all $i, j \in [n]$ (Schur multiplication).

Proposition

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $A \in \mathbb{M}(n \times n; \mathbb{F})$ be positive semidefinite. Then

$$\|S_A\| = \max\{A_{ii} : i \in [n]\}.$$

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.
Put $\alpha := \max\{A_{ii} : i \in [n]\}$.

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.
Put $\alpha := \max\{A_{ii} : i \in [n]\}$. Firstly, we recognise that

$$\|A * E_n\| = \|\mathrm{diag}(A)\| = \alpha,$$

implying that $\alpha = \|A * E_n\| \leq \|S_A\|$.

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.

Put $\alpha := \max\{A_{ii} : i \in [n]\}$. Firstly, we recognise that

$$\|A * E_n\| = \|\text{diag}(A)\| = \alpha,$$

implying that $\alpha = \|A * E_n\| \leq \|S_A\|$. Next, observe that both, the matrix $\alpha E_n - A * E_n$ and the block matrix $\begin{pmatrix} A & A \\ A^* & A \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ are positive semidefinite (why?).

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.
Put $\alpha := \max\{A_{ii} : i \in [n]\}$. Firstly, we recognise that

$$\|A * E_n\| = \|\text{diag}(A)\| = \alpha,$$

implying that $\alpha = \|A * E_n\| \le \|S_A\|$. Next, observe that both, the matrix $\alpha E_n - A * E_n$ and the block matrix $\begin{pmatrix} A & A \\ A^* & A \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ are positive semidefinite (why?). Let $B \in \mathbb{M}(n \times n; \mathbb{F})$ so that $\|B\| \le 1$.

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.

Put $\alpha := \max\{A_{ii} : i \in [n]\}$. Firstly, we recognise that

$$\|A * E_n\| = \|\text{diag}(A)\| = \alpha,$$

implying that $\alpha = \|A * E_n\| \leq \|S_A\|$. Next, observe that both, the matrix $\alpha E_n - A * E_n$ and the block matrix $\begin{pmatrix} A & A \\ A^* & A \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ are positive semidefinite (why?). Let $B \in \mathbb{M}(n \times n; \mathbb{F})$ so that $\|B\| \leq 1$. Then

$$\begin{pmatrix} \alpha E_n & A * B \\ (A * B)^* & \alpha E_n \end{pmatrix}$$

Norm of the Schur multiplier of a positive semidefinite matrix II

Proof.
Put $\alpha := \max\{A_{ii} : i \in [n]\}$. Firstly, we recognise that

$$\|A * E_n\| = \|\text{diag}(A)\| = \alpha,$$

implying that $\alpha = \|A * E_n\| \leq \|S_A\|$. Next, observe that both, the matrix $\alpha E_n - A * E_n$ and the block matrix $\begin{pmatrix} A & A \\ A^* & A \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ are positive semidefinite (why?). Let $B \in \mathbb{M}(n \times n; \mathbb{F})$ so that $\|B\| \leq 1$. Then

$$\begin{pmatrix} \alpha E_n & A * B \\ (A * B)^* & \alpha E_n \end{pmatrix} \overset{(!)}{=} \begin{pmatrix} \alpha E_n - A * E_n & 0 \\ 0 & \alpha E_n - A * E_n \end{pmatrix} + \begin{pmatrix} A * E_n & A * B \\ A * B^* & A * E_n \end{pmatrix}$$

Norm of the Schur multiplier of a positive semidefinite matrix III

Proof ctd.

Consequently,

$$\begin{pmatrix} \alpha E_n & A * B \\ (A * B)^* & \alpha E_n \end{pmatrix}$$

Norm of the Schur multiplier of a positive semidefinite matrix III

Proof ctd.

Consequently,

$$\begin{pmatrix} \alpha E_n & A * B \\ (A * B)^* & \alpha E_n \end{pmatrix} = \begin{pmatrix} \alpha E_n - A * E_n & 0 \\ 0 & \alpha E_n - A * E_n \end{pmatrix} + \begin{pmatrix} A & A \\ A & A \end{pmatrix} * \begin{pmatrix} E_n & B \\ B^* & E_n \end{pmatrix}$$

is a sum of two positive semidefinite matrices (why?) and hence positive semidefinite, too.

Norm of the Schur multiplier of a positive semidefinite matrix III

Proof ctd.

Consequently,

$$\begin{pmatrix} \alpha E_n & A * B \\ (A * B)^* & \alpha E_n \end{pmatrix} = \begin{pmatrix} \alpha E_n - A * E_n & 0 \\ 0 & \alpha E_n - A * E_n \end{pmatrix} + \begin{pmatrix} A & A \\ A & A \end{pmatrix} * \begin{pmatrix} E_n & B \\ B^* & E_n \end{pmatrix}$$

is a sum of two positive semidefinite matrices (why?) and hence positive semidefinite, too. Due to the last corollary (to Pisier's Theorem applied to the block matrix case) it follows that $\|A * B\| \leq \alpha$. Hence, $\|S_A\| \leq \alpha$, and the claim follows. $\square$

Norm of the Schur multiplier of a positive semidefinite matrix IV

It is well-known that in general $\|S_B\| \leq \|B\|$ for all $B \in \mathbb{M}(n \times n; \mathbb{F})$. Thus, another interesting observation is the following one:

Norm of the Schur multiplier of a positive semidefinite matrix IV

It is well-known that in general $\|S_B\| \leq \|B\|$ for all $B \in \mathbb{M}(n \times n; \mathbb{F})$. Thus, another interesting observation is the following one:

Corollary

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $B \in \mathbb{M}(n \times n; \mathbb{F})$. TFAE

(i) $\begin{pmatrix} \|S_B\| E_n & B \\ B^* & \|S_B\| E_n \end{pmatrix}$ *is positive semidefinite.*

(ii) $\|S_B\| = \|B\|$.