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Positivity

# Open problems

One of the most fruitful activities that takes place at a conference is the interchange of problems. Often, in later years the success of a conference may be judged by the number of problems raised there which are eventually solved. The papers in the remainder of this volume contain several problems, but here are some more that did not fit naturally into any of those papers. I hope to see solutions of at least some of these problems in the pages of *Positivity* in years to come.

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## 1. Some questions about operators on Banach lattices

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Let $E$ and $F$ be Banach lattices, $T, K \in \mathcal{L}(E, F)$ with $0 \leq T \leq K$. We have shown that if either $E'$ or $F$ has an order continuous norm, $T$ is an almost interval-preserving or lattice homomorphism and $K$ is compact then $T$ is compact.

Similarly if $E$ and $F$ are Banach lattices, $T, U \in \mathcal{L}(E, F)$ with $0 \leq T \leq U$ with $U$ being weakly compact and $T$ is either almost interval-preserving or a lattice homomorphism, then $T$ is weakly compact.

**Problem 1.1.** Is a similar result true for Dunford-Pettis operators? To be precise, if $E$ and $F$ are Banach lattices, $T, U \in \mathcal{L}(E, F)$ with $0 \leq T \leq U$, $U$ is a Dunford-Pettis operator and $T$ is either almost interval-preserving or a lattice homomorphism, then is $T$ necessarily a Dunford-Pettis operator?

We would like to ask the following two concrete questions concerning the density of regular operators in continuous operators.

**Problem 1.2.** For $1 < p < q < \infty$, is $\mathcal{L}^r(\ell_p, \ell_q)$ dense in $(\mathcal{L}(\ell_p, \ell_q), \| \cdot \|)$?

**Problem 1.3.** For $1 < p < q < \infty$, is $\mathcal{L}^r(L^p[0,1], L^q[0,1])$ dense in $(\mathcal{L}(L^p[0,1], L^q[0,1], \| \cdot \|)$?

## 2. A problem on direct sum decompositions

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In a major forthcoming work by Dales and M.E. Polyakov on multi-norms, they study Banach lattices as important special examples. The theory would be simplified by a positive answer to the following question, which does not appear to be one that would arise naturally when considering Banach lattices from a real viewpoint.

**Problem 2.1.** Suppose that $E$ is a complex Banach lattice with an algebraic direct sum decomposition $E = F \oplus G$ which satisfies the property $\| |x| \vee |y| \| = \|x + y\|$ for all $x \in F$ and $y \in G$. Is $F \perp G$?

Note that for real scalars this fails, as may be seen by taking $E = \mathbb{R}^2$ with the supremum norm, $F = \{(x, x) : x \in \mathbb{R}\}$ and $G = \{(y, -y) : y \in \mathbb{R}\}$.

## 3. Some open problems in the geometry of operator ideals

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### The principle of local reflexivity for maximal Banach ideals

We essentially adopt notation and terminology from [15], which however cannot be explained here in detail, due to the limitation of space. Therefore, we would like to refer the interested reader to [2,6,15], the excellent survey article [3] and the further references therein.

Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary *maximal* Banach ideal. It seems to be *still* an open problem whether it is *always* possible to transfer the norm estimation in the far-reaching classical principle of local reflexivity to the ideal norm $\mathbf{A}$. If this were not the case, we would be very interested in constructing an explicit counterexample. More precisely, we would like to know whether there exists a *maximal* Banach ideal $(\mathfrak{A}_0, \mathbf{A}_0)$ which does *not* satisfy the following factorization property:

**Definition 3.1.** [cf. [9,10]] *Let $E$ and $Y$ be Banach spaces, where $E$ is finite dimensional and $F$ is a finite dimensional subspace of $Y'$. Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal and $\varepsilon > 0$. We say that the principle of $\mathfrak{A}$-local reflexivity (short: $\mathfrak{A}$-LRP) is satisfied, if for every $T \in \mathfrak{L}(E, Y'')$ there exists an operator $S \in \mathfrak{L}(E, Y)$ such that*

(i) $\mathbf{A}(S) \leq (1+\epsilon) \cdot \mathbf{A}(T)$;
(ii) $\langle Sx, y' \rangle = \langle y', Tx \rangle$ for all $(x, y') \in E \times F$;
(iii) $j_Y Sx = Tx$ *for all* $x \in T^{-1}(j_Y(Y))$.[1]

Obviously, the classical principle of local reflexivity is simply the $\mathfrak{B}$-LRP, where $(\mathfrak{B}, \mathbf{B}) := (\mathfrak{L}, \|\cdot\|)$.

**Problem 3.2.** Does every maximal Banach ideal satisfy the $\mathfrak{A}$-LRP?

Due to the finite nature of *maximal* Banach ideals and the $\mathfrak{A}$-LRP, local versions of injectivity and surjectivity of operator ideals play a key role. Moreover, they imply interesting results for operators with infinite dimensional range (cf. [11–13]). These are the following so-called "accessibility conditions", treated in detail in [1,2]:

**Definition 3.3.** *Let* $(\mathfrak{B}, \mathbf{B})$ *be a* $p$*-normed Banach ideal, where* $0 < p \leq 1$.

(i) $(\mathfrak{B}, \mathbf{B})$ *is called right-accessible, if for any two Banach spaces* $E$ *and* $Y$, *dim*$(E) < \infty$, *any operator* $T \in \mathfrak{L}(E, Y)$ *and any* $\varepsilon > 0$ *there are a finite dimensional subspace* $N$ *of* $Y$ *and* $S \in \mathfrak{L}(E, N)$ *such that* $T = J_N^Y S$ *and* $\mathbf{B}(S) \leq (1+\varepsilon)\mathbf{B}(T)$; *here* $J_N^Y : N \hookrightarrow Y$ *denotes the canonical embedding.*
(ii) $(\mathfrak{B}, \mathbf{B})$ *is called left-accessible, if for any two Banach spaces* $F$ *and* $X$, *dim*$(F) < \infty$, *any operator* $T \in \mathfrak{L}(X, F)$ *and any* $\varepsilon > 0$ *there are a finite codimensional subspace* $L$ *of* $X$ *and* $S \in \mathfrak{L}(X/L, F)$ *such that* $T = SQ_L^X$ *and* $\mathbf{B}(S) \leq (1+\varepsilon)\mathbf{B}(T)$; *here* $Q_L^X : X \twoheadrightarrow X/L$ *denotes the canonical quotient map.*
(iii) $(\mathfrak{B}, \mathbf{B})$ *is called accessible, if it is both, right-accessible and left-accessible.*

**Proposition 3.4 (cf. [13]).** *If* $(\mathfrak{A}, \mathbf{A})$ *is a right-accessible maximal Banach ideal, then the* $\mathfrak{A}$*-LRP holds.*

We *still* do not know whether the statement in Proposition 3.4 can be reversed. In 1993, Pisier constructed explicitly a maximal Banach ideal $(\mathfrak{A}_P, \mathbf{A}_P)$ which is not right-accessible, and solved — 37 years later only — a further problem of Grothendieck's famous Résumé (cf. [5] and [2]). Consequently, Problem 3.2 is even harder than Grothendieck's tough accessibility problem. A maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ which does not satisfy the $\mathfrak{A}$-LRP, necessarily cannot be right-accessible! Notice that in the investigation of those problems, Banach spaces *without* the approximation property (such as the Pisier space $P$) are necessarily involved (cf. [2,9,13]). Consequently, we arrive at

**Problem 3.5.** Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. Are then the following statements equivalent?

(i) $(\mathfrak{A}, \mathbf{A})$ is right-accessible.
(ii) $(\mathfrak{A}, \mathbf{A})$ satisfies the $\mathfrak{A}$-LRP.

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[1] Here, $j_Y : Y \hookrightarrow Y''$ denotes the canonical embedding from the Banach space $Y$ into its bidual $Y''$.

**Problem 3.6.** Does $(\mathfrak{A}_P, \mathbf{A}_P)$ satisfy the $\mathfrak{A}_P$-LRP?

Let $(\mathfrak{B}, \mathbf{B})$ be a $p$-normed Banach ideal, where $0 < p \leq 1$. In 1971, Pietsch defined the adjoint operator ideal $(\mathfrak{B}^*, \mathbf{B}^*)$ to uncover the structure of maxi-mal Banach ideals. In 1973, Gordon-Lewis-Retherford introduced a related smaller "conjugate" operator ideal $(\mathfrak{B}^\Delta, \mathbf{B}^\Delta)$, which – in the contrary – is somehow "skew" under the point of view of trace duality but nevertheless quite useful (cf. [4,9]):

**Definition 3.7.** *Let $X$, $Y$ two Banach spaces and $\mathfrak{B}^\Delta(X, Y)$ be the set of all those operators $T \in \mathfrak{L}(X, Y)$ satisfying*

$$\mathbf{B}^\Delta(T) := \sup\{|\mathrm{tr}(TL)| : L \in \mathfrak{F}(Y, X), \mathbf{B}(L) \leq 1\} < \infty.$$

*Then $(\mathfrak{B}^\Delta, \mathbf{B}^\Delta)$ is a Banach ideal.*[2] *It is called the conjugate ideal of $(\mathfrak{B}, \mathbf{B})$.*

It is easy to see that the various approximation properties of Banach spaces and the accessibility of operator ideals are intrinsically related to a "good behaviour" of trace duality. If $\mathfrak{A}$ is a maximal Banach ideal, then $(\mathfrak{A}^\Delta, \mathbf{A}^\Delta)$ is right-accessible (cf. [9,10]). However, we do not know whether $\mathfrak{A}^\Delta$ is left-accessible, since:

**Theorem 3.8 (cf. [9,10]).** *Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. Then the following statements are equivalent:*

(i) *$\mathfrak{A}^\Delta$ is left–accessible.*
(ii) *$\mathfrak{A}(E, Y'') \stackrel{1}{=} \mathfrak{A}(E, Y)''$ for all finite dimensional Banach spaces $E$ and arbitrary Banach spaces $Y$.*
(iii) *The $\mathfrak{A}$-LRP holds.*

**Property (I) and normed products of Banach ideals**

After extending finite rank operators in suitable quasi-Banach ideals $\mathfrak{B} \neq \mathfrak{L}$, the $\mathfrak{B}$-LRP and the calculation of conjugate ideal-norms allow us to ignore the structure of the range space and to leave the finite dimensional case. There are sufficient conditions on $\mathfrak{B}$ which imply that for any two Banach spaces $X$ and $Y$, any Banach space $Z$ which contains $X$ isometrically, any finite rank operator $L \in \mathfrak{F}(X, Y)$ and any $\varepsilon > 0$ there exists a *finite rank extension* $\widetilde{L} \in \mathfrak{F}(Z, Y)$ such that $\mathbf{B}(\widetilde{L}) \leq (1 + \epsilon) \cdot \mathbf{B}(L)$ (cf. [13]).[3]

Consequently, we have to look for a suitable class of product ideals $\mathfrak{A} \circ \mathfrak{B}$ which satisfy the following factorization property: Given $\varepsilon > 0$, any finite rank operator $L \in \mathfrak{A} \circ \mathfrak{B}$ can be factorized as $L = AB$ such that $\mathbf{A}(A) \cdot \mathbf{B}(B) \leq (1 + \varepsilon) \cdot \mathbf{A} \circ \mathbf{B}(L)$ and $A$ respectively $B$ has finite dimensional range. Product ideals $\mathfrak{A} \circ \mathfrak{B}$ of this type are said to have *property (I)* respectively *property (S)*. They had been introduced in [7] to prepare a detailed investigation of trace ideals.

[2] Here $\mathrm{tr} : \mathfrak{F}(Y, Y) \longrightarrow \mathbb{K}$ denotes the trace on the operator ideal component of all finite rank operators on $Y$.

[3] Recall that there is no Hahn-Banach extension theorem for finite rank operators, viewed as elements of the Banach ideal $(\mathfrak{L}, \|\cdot\|)$.

In looking for a counterexample of a maximal Banach ideal $(\mathfrak{A}_0, \mathbf{A}_0)$ which does not satisfy the $\mathfrak{A}_0$-LRP, property (I) of product ideals of type $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ seemingly plays a key role.

**Theorem 3.9 (cf. [13]).** *Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal such that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has property $(I)$. If* space$(\mathfrak{A})$ *contains a Banach space $X_0$ such that $X_0$ has the bounded approximation property but $X_0''$ has not, then the $\mathfrak{A}^*$-LRP is not satisfied.*

A further interesting class is given by the family of all operator ideals which contain $(\mathfrak{L}_2, \mathbf{L}_2)$ as a factor (i. e., the maximal and injective Banach ideal of all operators which factor through a Hilbert space). This class does not only show surprising connections with the principle of local reflexivity for operator ideals. There are also links to the existence of an ideal-*norm* on certain product ideals and the property (I), reflected e. g. in the following two results (cf. [14]):

**Theorem 3.10.** *Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary maximal Banach ideal such that $\mathfrak{A} \circ \mathfrak{L}_2$ is normed. If the $(\mathfrak{A} \circ \mathfrak{L}_2)^{**}$-LRP is satisfied, then $(\mathfrak{A} \circ \mathfrak{L}_2)^{**} \circ \mathfrak{L}_\infty$ has both, property $(I)$ and property $(S)$.*

**Theorem 3.11.** *Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary maximal Banach ideal such that $\mathfrak{A} \circ \mathfrak{L}_2$ is normed. If $\mathfrak{L}_2 \subseteq \mathfrak{A}^*$, then the $(\mathfrak{A} \circ \mathfrak{L}_2)^{**}$-LRP cannot be satisfied.*

Here is a nice application of Theorem 3.11: Put $\mathfrak{A} := (\mathfrak{L}_1 \circ \mathfrak{L}_\infty)^{**}$. Due to Grothendieck's inequality in operator form, it follows that $\mathfrak{L}_2 \subseteq \mathfrak{A}^*$ (cf. [2,12]). Hence, in view of Theorem 3.11, a natural question appears:

**Problem 3.12.** Is the product ideal $(\mathfrak{L}_1 \circ \mathfrak{L}_\infty)^{**} \circ \mathfrak{L}_2$ normed?

However, we do not know criteria which are sufficient for the existence of an ideal-*norm* on a given product of quasi-Banach ideals. Here, one has to study very carefully the structure of $F$-spaces which are not locally convex, such as $L^p([0,1])$ and the Hardy space $H^p$ on the unit disk, for some $0 < p < 1$ (cf. [8]). It seems to be much easier to provide arguments which imply the non-existence of such an ideal-norm (by using trace ideals) (cf. [14]).

**Theorem 3.13 (cf. [12,13]).** *Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal such that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ is right-accessible and has property $(I)$. Let $X$ and $Y$ be Banach spaces such that $X'$ and $Y$ are of cotype 2. If the $\mathfrak{A}^*$-LRP is satisfied, then*

$$\mathfrak{A}^{\mathrm{inj}}(X, Y) \subseteq \mathfrak{L}_2(X, Y)$$

*and*

$$\mathbf{L}_2(T) \leq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^{\mathrm{inj}}(T)$$

*for all operators $T \in \mathfrak{A}^{\mathrm{inj}}(X, Y)$.*

**Corollary 3.14.** *Let $(\mathfrak{B}, \mathbf{B})$ be a maximal Banach ideal. Let $X_0$ and $Y_0$ be Banach spaces such that $X_0'$ and $Y_0$ are of cotype 2 and $\mathfrak{B}^*(X_0, Y_0) \not\subseteq \mathfrak{L}_2(X_0, Y_0)$. If $\mathfrak{B} \circ \mathfrak{L}_\infty$ is right-accessible and has property $(I)$, then the $\mathfrak{B}$-LRP is not satisfied.*

Consequently, an application of Proposition 3.4 immediately implies the following surprising result:

**Corollary 3.15.** *Let* $(\mathfrak{B}, \mathbf{B})$ *be a maximal Banach ideal. Let* $X_0$ *and* $Y_0$ *be Banach spaces such that* $X_0'$ *and* $Y_0$ *are of cotype 2 and* $\mathfrak{B}^*(X_0, Y_0) \not\subseteq \mathfrak{L}_2(X_0, Y_0)$*. If* $\mathfrak{B}$ *is right-accessible, then* $\mathfrak{B} \circ \mathfrak{L}_\infty$ *does not have property* $(I)$*.*

**Problem 3.16.** Let $(\mathfrak{C}_2, \mathbf{C}_2)$ denote the maximal injective Banach ideal of all cotype 2 operators. Does $\mathfrak{C}_2^* \circ \mathfrak{L}_\infty$ have property $(I)$?

If Problem 3.16 had a positive answer, Corollary 3.15 would imply that $(\mathfrak{C}_2, \mathbf{C}_2)$ cannot be left-accessible; and a further open problem of Defant and Floret could be solved (see [2], 21.2., p. 277).

**Problem 3.17.** Can we drop the assumption "$\mathfrak{B} \circ \mathfrak{L}_\infty$ is right-accessible" in Corollary 3.14?

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## 4. Problems on nonnegativity of Moore–Penrose inverses

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Let $H, H_1, H_2$ denote Hilbert spaces over $\mathbb{R}$. $\mathcal{B}(H_1, H_2)$ denotes the set of all bounded linear operators from $H_1$ into $H_2$. When $H = H_1 = H_2$, $\mathcal{B}(H_1, H_2)$ will be denoted by $\mathcal{B}(H)$.

A subset $K$ of a Hilbert space $H$ is called a cone if, (i) $x,\ y \in K \Longrightarrow x+y \in K$ and (ii) $x \in K, \alpha \in \mathbb{R}, \alpha \geq 0 \Longrightarrow \alpha x \in K$. Let $K$ be a cone. The dual cone of $K$ denoted $K^*$ is defined as $K^* = \{x \in H \colon \langle x, t\rangle \geq 0,\ \forall\ t \in K\}$. A cone $C$ is said to be *acute* if $\langle x, y\rangle \geq 0$, for all $x, y \in C$. $C$ is said to be *obtuse* if $C^* \cap \{cl\ span\ C\}$ is acute, where *span* $C$ denotes the linear subspace spanned by $C$. In particular, if $A \in \mathcal{B}(H_1, H_2),\ K \subseteq H_1$, a closed cone with $C = AK$, then the obtuseness of $C$ is defined to be the acuteness of $(AK)^* \cap\ R(A)$.

Let $A \in \mathcal{B}(H_1, H_2)$ be with closed range. Let $A^*$ denote the adjoint of $A$. The Moore–Penrose inverse of $A$ is the unique operator $A^\dagger$ in $\mathcal{B}(H_2, H_1)$ which satisfies the following equations: $AA^\dagger A = A$; $A^\dagger AA^\dagger = A^\dagger$; $(AA^\dagger)^* = AA^\dagger$; $(A^\dagger A)^* = A^\dagger A$. Let $A \in \mathcal{B}(H)$. The group inverse of $A$ is the unique operator $A^\#$ in $\mathcal{B}(H)$ , if it exists, that satisfies the equations: $AA^\# A = A$; $A^\# AA^\# = A^\#$; $AA^\# = A^\# A$. It is well known that the group inverse exists if and only if $R(A)$ and $N(A)$ are complementary subspaces, [2]. In particular, it follows that if $A$ is hermitian, then $A^\#$ exists.

Recently, the following characterization of nonnegativity of the operator $(A^*A)^\dagger$ was proved.

**Theorem 4.1.** *(Theorem 3.6, [1]) Let $A \in \mathcal{B}(H_1, H_2)$ with $R(A)$ closed, $K$ be a closed convex cone of $H_1$ with $A^\dagger AK \subseteq K$. Let $C = AK$ and $D = (A^\dagger)^* K^*$. Then the following conditions are equivalent:*

(a) $(A^*A)^\dagger(K^*) \subseteq K$.
(b) $C^\circ \cap R(A) \subseteq -C$.
(c) *$D$ is acute.*
(d) *$C$ is obtuse.*
(e) $A^*Ax \in P_{R(A^*)}(K^*),\ x \in R(A^*) \Longrightarrow x \in K$.
(f) $A^*Ax \in K^*,\ x \in R(A^*) \Longrightarrow x \in K$.

The present author also obtained the following necessary and sufficient conditions for the nonnegativity of $(A^*A)^\dagger$. This includes a generalization of a classical result of Novikoff on nonnegative inverses of the Gram operator $A^*A$. (see the references cited in [3]).

**Theorem 4.2.** *(Theorem 3.17, [3]) Let $A \in \mathcal{B}(H_1, H_2)$ with $R(A)$ closed, $K$ be a closed cone of $H_1$ with $A^\dagger AK \subseteq K$. Let $C = AK$ and $D = (A^\dagger)^* K^*$. Then the following conditions are equivalent:*

(a) *$D$ is acute.*
(b) $(A^*A)^\dagger K^* \subseteq K + N(A)$.
(c) *$C$ is obtuse.*
(d) *For every $x \in H_2$, there exists $y \in H_2$ such that $y \pm x \in AK$ and $\| y \| \leq \| x \|$ .*

We observe that the inclusion $(A^*A)^\dagger K^* \subseteq K + N(A)$ does in fact imply nonnegativity of $(A^*A)^\dagger$, as proved in Lemma 3.18, [3].

We now pose the following questions which appear to be nontrivial and interesting:

**Problem 4.3.** To obtain a result that combines Theorem 4.1 and Theorem 4.2, and which could also include other equivalent conditions.

**Problem 4.4.** To extend characterizations of nonnegativity of the Moore–Penrose inverse of an operator $A$ rather than $A^*A$, as in Theorem 4.1 or Theorem 4.2 (or both) in such a way that these very same results follow as special cases.

**Problem 4.5.** Same as problem 4.4, with the group inverse in place of the Moore–Penrose inverse. (Note that the group inverse of $A^*A$ always exists).

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