MATHEMATICS

Proof of Ultrafilter lemma with two propositions and Zorn lemma

Asked 6 years, 8 months ago Modified 6 years, 8 months ago Viewed 662 times

I would like to prove the following:

> Let X be any set, then every filter $\mathcal{F}$ on X is contained in an ultrafilter F

Using **two propositions** and **Zorn Lemma**. I am required to come up with the propositions and the proof. Can someone check if everything looks okay.

Recall definitions:

A filter $\mathcal{F} \subset \mathcal{P}(X)$ is a collection of sets satisfying:

> 1. $\varnothing \notin \mathcal{F}$
> 2. $A \in \mathcal{F}, A \subseteq B \implies B \in \mathcal{F}$
> 3. $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$

and

> F is an ultrafilter if for all $\mathcal{F}$ a filter on X, $F \not\subset \mathcal{F}$

Proposition 1: If F is a maximal element of $(\mathbb{F}, \subseteq)$, where $\mathbb{F} = \{\mathcal{F} | \mathcal{F} \text{ is a filter on } X\}$, then $\mathcal{F} \subset F, \forall \mathcal{F} \in \mathbb{F}$ *(Read: Maximal element of poset* $(\mathbb{F}, \subseteq)$ *contains all filters)*

Proposition 2: Every chain in $\mathbb{F}$ given above has an upperbound given by $G = \bigcup \mathcal{C}$, $\mathcal{C}$ is a chain of filters.

Zorn Lemma (Adapted to this scenario): Given $(\mathbb{F}, \subseteq)$ a poset, $\subseteq$ is ordering by inclusion, if every chain in $\mathbb{F}$ has an upperbound, then $\mathbb{F}$ contains a maximal element.

proof:

Proof of proposition 1: Let F be the maximal element on $(\mathbb{F}, \subseteq)$, take a filter $\mathcal{F} \in \mathbb{F}$, we wish to show that $\mathcal{F} \subseteq F$. By contradiction, suppose $\mathcal{F} \nsubseteq F$, then either $F \subset \mathcal{F}$ or $F \cap \mathcal{F} = \varnothing$.

We eliminate the former case because it contradictions the definition of being maximal. Suppose $F \cap \mathcal{F} = \varnothing$, take $A \in \mathcal{F}$, then by definition of being maximal, $A \in F$ or $X \backslash A \in F$. By assumption, A cannot be in F, therefore for all $A \in \mathcal{F}, X \backslash A \in F$. Then suppose $A = X$, then $X \backslash X = \varnothing \in F \implies$ contradicts definition of being a filter.

Therefore, F contains all $\mathcal{F}$ a filter on X.

Proof of proposition 2: Let $\mathcal{C}$ be a chain on $\mathbb{F}$, then $\mathcal{C} = \{\mathcal{F}_\alpha | \alpha \in I\}$ for some index set I. Take $G = \bigcup \mathcal{C}$ and we claim this is the upperbound of $\mathcal{C}$. Indeed, $\forall \mathcal{F}_\alpha \in \mathcal{C}, \mathcal{F}_\alpha \subset G$. We

wish to show that G is a filter. Suppose not, then there exists $A \in G$ such that $A \subset B$, $B \subset \mathcal{P}(X)$ and $B \notin G$. Since A is contained in some $\mathcal{F}_i \subset G$, then by definition of a chain, $\exists k > i$, such that $\mathcal{F}_i \subset \mathcal{F}_k$ and $A \in \mathcal{F}_k$, $k > i$. Then $\forall B$, $B \subset \mathcal{P}(X)$
$A \subset B, \implies B \in \mathcal{F}_k$. But since $G = \bigcup \mathcal{C}$, $B \in G$. Contradiction. So $G = \bigcup \mathcal{C}$ is a filter.

Conclusion by Zorn Lemma: We know that given $(\mathbb{F}, \subseteq)$ a poset, $\subseteq$ is ordering by inclusion, if every chain in $\mathbb{F}$ has an upperbound, then $\mathbb{F}$ contains a maximal element. By (proposition 2), every chain has an upperbound, then $\mathbb{F}$ contains a maximal element F which is an ultrafilter on X and by (proposition 1) contains every filter on X. So every set X has an ultrafilter that contains every filter on X

Can someone check the above is sound. Please let me know if there is need for corrections. Thank you.

elementary-set-theory | proof-verification | order-theory | axiom-of-choice | filters

Share Cite Follow

edited Apr 13, 2017 at 12:21
Community Bot
1

asked Jul 11, 2016 at 13:40

Shamisen Expert
7,662 7 41 85

You did not check closure of G under intersection. – André Nicolas Jul 11, 2016 at 13:54

Proposition 1 is false. If it were true, there could only be one ultrafilter on any set. Moreover, your final argument, even if it were correct, would only establish the existence of some ultrafilter on X, not one containing a given filter $\mathscr{F}$. – Brian M. Scott Jul 11, 2016 at 14:01

The definition of ultrafilter I am accustomed to builds in for all A, A is in the ultrafilter D or $X \setminus A$ is. Extensions of filter to ultrafilter are often highly non-unique, so one should not say or assume that there is **the** maximal element. – André Nicolas Jul 11, 2016 at 14:19

@AndréNicolas Hi from the feedbacks I am not certain what I should change. That being said, I could add in closure of G under intersection and also check for $\varnothing$ case. But I am not sure how to fix proposition 1. – Shamisen Expert Jul 11, 2016 at 14:27

2 (More) So suppose neither is. Show that we can add B to U to obtain a filter U' that properly extends U. We define U' as the collection of all supersets of sets $B \cap Y$, where Y ranges over U. The only way this can fail to be a filter is if it contains the empty set, but then $X \setminus B$ is in U. – André Nicolas Jul 11, 2016 at 14:51

1 Answer

Sorted by: Highest score (default)

2

You're given a filter $\mathscr{F}$ on X, and you want to show that there's an ultrafilter $\mathscr{U}$ on X such that $\mathscr{F} \subseteq \mathscr{U}$. Since you're going to be using Zorn's lemma, it's certainly reasonable to suppose that $\mathscr{U}$ will be a maximal element some partial order, but you need to make sure that this maximal element contains $\mathscr{F}$. The easiest way to do that is to start with a partial order of objects that contain $\mathscr{F}$: let

$$\mathbb{P} = \{\mathscr{G} \subseteq \wp(X) : \mathscr{F} \subseteq \mathscr{G} \text{ and } \mathscr{G} \text{ is a filter on } X\} \; ,$$

and consider the partial order $\langle \mathbb{P}, \subseteq \rangle$. Any maximal element of $\mathbb{P}$ will at least be a filter on X containing X.

In order to apply Zorn's lemma, you'll have to show that $\langle \mathbb{P}, \subseteq \rangle$ satisfies its hypothesis, i.e., that every chain in the partial order has an upper bound in $\mathbb{P}$; this is essentially your **Proposition 2** with a slightly different partial order.

In order for Zorn's lemma to give you the desired result, you need to prove that a maximal element of $\mathbb{P}$ is an ultrafilter on X. With your definition of *ultrafilter* there's nothing to prove, and I have no idea what second proposition you're expected to come up with. If you were using the other common definition, mentioned by **André Nicolas** in the comments, that a filter $\mathscr{U}$ on X is an ultrafilter if and only if for each $A \subseteq X$, exactly one of A and $X \setminus A$ belongs to $\mathscr{U}$, you *would* need a second proposition:

Proposition. If $\mathscr{U}$ is a filter on X that is maximal with respect to $\subseteq$, then $\mathscr{U}$ is an ultrafilter.

answered Jul 11, 2016 at 15:35
Brian M. Scott
601k 55 738 1213