Stochastic Modelling of Counterparty Credit Risk and CVA, with a questioning View towards the New Regulatory Framework

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–

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This presentation and associated materials are provided for informational and educational purposes only. Views expressed in this work are the authors' views. They are not necessarily shared by the Federal Financial Supervisory Authority.

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In particular, our research is by no means linked to any present and future wording regarding global regulation of CCR including EMIR and CRR.

Contents

On CCR and its modelling challenges I

Suppose there are two parties who are trading a portfolio of OTC derivative contracts such as, e. g., a portfolio of CDSs. Counterparty credit risk (CCR) is the risk that *at least one* of those two parties in that derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments to its counterpart.

On CCR and its modelling challenges II

- CCR is of bilateral nature, based on a contractual exchange of cashflows between two parties over a period of time.
- Future cashflow exchanges are not known with certainty today. The main feature that distinguishes CCR from the risk of a standard loan is the uncertainty of the exposure at any future date. Hence, regarding the modelling of the exposure a simulation of future cashflow exchanges is necessary (Nested Monte-Carlo, SDEs, PDEs, SPDEs, grid computing . . .).
- Wrong-Way/Right-Way Risk (WWR/RWR): strong relationship between credit risk and market risk. So, we need a truly dynamic (portfolio) credit risk model for both parties: static copula models are not enough. Default intensities should depend on systematic economic factors!

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- Let $k \in \{0, 2\}$ be given and let $X = (X(t))_{0 \leq t \leq T}$ denote a stochastic process describing a cash flow between party 0 and party 2 (or a random sequence of prices). If, at time t, X_t is seen from the point of view of party k, we denote its value equivalently as $X_t(k)$ or $X_t(k; 2-k)$ or $X_k(t)$ - depending on its eligibility.

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- Moreover, we will make use of the important notation $Y_t(k \,|\, 2 - k)$ to describe a cash flow Y from the point of view of party k at time t contingent on the default of party $2 - k$.

Information setup

Let τ_0 denote the default time of the investor and τ_2 the default time of the counterparty. Suppose that the underlying financial market model is arbitrage-free. Fix a filtered probability space $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{Q})$, satisfying the usual conditions. Let $\mathcal{G} := \bigvee_{t\geq 0} \mathcal{G}_t := \sigma\left(\bigcup_{t\geq 0} \mathcal{G}_t\right)$ where the σ-algebra $\mathcal{G}_t$ contains both, the market information up to time t and the information whether the default of the investor or its counterpart has occurred or not up to time t. $\mathbb{Q}$ is a (not necessarily unique) “spot martingale measure”. Fix $t \in \mathbb{R}_+$. Let the σ-algebra $\mathcal{F}_t$ represent the observable market information up to time t but the default event. $\mathcal{G}_t$ is then defined as

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t \supseteq \mathcal{F}_t$$

where $\mathcal{H}_t := \mathcal{H}_t^{(0)} \vee \mathcal{H}_t^{(2)}$ defines the filtration $\mathbf{H}$ generated by the default times, i. e., $\mathcal{H}_t^{(k)} := \sigma\left(H_u^{(k)}; 0 \leq u \leq t\right)$, where $H_u^{(k)} := \mathbb{1}_{\{\tau_k \leq u\}}$.

First default at $\tau^* = \min\{\tau_0, \tau_2, T\}$

Consider $\tau := \min\{\tau_0, \tau_2\}$ (i. e., the "first-to-default time"). By construction both, τ_0 and τ_2 are $\mathbf{G}$-stopping times.

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Consider the $\mathbf{G}$-stopping time $\tau^* := \min\{\tau, T\}$. We denote by $\mathbb{E}^{\mathbb{Q}}_{\tau^*}$ the conditional expectation under $\mathbb{Q}$ given the stopped filtration $\mathcal{G}_{\tau^*}$, i. e., $\mathbb{E}^{\mathbb{Q}}_{\tau^*} := \mathbb{E}^{\mathbb{Q}}[\,\cdot\,|\mathcal{G}_{\tau^*}]$. The CCR analysis is based on the functions $x^+ := \max\{x, 0\}$ and $x^- := x^+ - x = (-x)^+ = \max\{-x, 0\}$ $(x \in \mathbb{R})$.

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$N = \{\tau^* = T\} \cap \{\tau^* \neq \tau\}$ and $A_k^- = \{\tau_k = \tau^*\} \cap \{\tau^* \neq \tau_{2-k}\}$, implying that $N \in \mathcal{G}_{\tau^*}$ and $A_k^- \in \mathcal{G}_{\tau^*}$.

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In the following we assume that $A_{\text{sim}} = \emptyset$ $\mathbb{Q}$-a.s.

Vulnerable cash flows and money conservation

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Definition

Let $X \equiv (X_t)_{t\in[0,T]}$ be an arbitrary $\mathbf{G}$-adapted stochastic process. X is called non-vulnerable if X and $\mathbb{1}_N X$ $\mathbb{Q}$-almost surely have the same sample paths, else X is called vulnerable.

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Standing Assumption (Money Conservation Principle)

Let $0 \leq t \leq T$ and $k \in \{0, 2\}$. Any non-vulnerable cash flow $X = (X_t)_{0 \leq t \leq T}$ between party k and its counterpart $2-k$, resulting from a trade between party k and its counterpart $2-k$, satisfies

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Hence, a “sure” asset for party k represents a “sure” liability for party $2-k$.

Valuation of defaultable claims I

Defaultable claims can be valued by interpreting them as portfolios of claims between non-defaultable counterparties including the riskless claim and mutual default protection contracts. Fix $k \in \{0, 2\}$. From the point of view of party k latter says:

Party k sells to party $2 - k$ default protection on party $2 - k$ contingent to an amount specified by an ISDA close-out rule.

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Let $0 \leq t < \tau^*$ ($\mathbb{Q}$-a.s.) and let

- $M_t(k)$ be the mark-to-market value to party k in case both, party k and party $2 - k$ do not default until T;
- $CVA_t(k \,|\, 2 - k)$ be the value of default protection that party k sells to party $2 - k$ contingent on the default of party $2 - k$.

Valuation of defaultable Claims II

At t party k requires a payment of the “CCR risk premium” $CVA_t(k \,|\, 2-k) > 0$ from party $2-k$ to be compensated for the risk of a default of party $2-k$.

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At t party k requires a payment of the "CCR risk premium" $CVA_t(k \,|\, 2-k) > 0$ from party $2-k$ to be compensated for the risk of a default of party $2-k$. Conversely, party $2-k$ requires a payment of $CVA_t(2-k \,|\, k) > 0$ from party k to be compensated for the risk of a default of party k. Therefore, party $2-k$ reports at t the "bilaterally CCR-adjusted" value (defined as "fair value" in FAS 157):

$$V_t(2-k) := -CVA_t(2-k \,|\, k) + M_t(2-k) + CVA_t(k \,|\, 2-k)$$

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The inclusion of DVA began 2005. In September 2006 the accounting standard in relation to fair value measurements FAS 157 (*The Statements of Financial Accounting Standard, No 157*) asked banks to record a DVA entry (implying that the DVA of one party is the CVA of the other).

Valuation of defaultable Claims III

FAS 157 namely says: "... *Because nonperformance risk includes the reporting entity's credit risk, the reporting entity should consider the effect of its credit risk (credit standing) on the fair value of the liability in all periods in which the liability is measured at fair value under other accounting pronouncements*..."

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The European equivalent of FAS 157 is the fair value provision of IAS 39 which had been published by the International Accountancy Standards Board in 2005, showing similar wording with respect to the valuation of CCR.

So, we define

$$DVA_t(k; 2-k) := CVA_t(2-k \mid k)\,.$$

Valuation of defaultable Claims IV

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Consequently,

$$V_t(2-k) \quad = \quad -CVA_t(2-k\,|\,k) + M_t(2-k) + CVA_t(k\,|\,2-k)$$

Valuation of defaultable Claims IV

Consequently,

$$\begin{aligned} V_t(2-k) &= -CVA_t(2-k\,|\,k) + M_t(2-k) + CVA_t(k\,|\,2-k) \\ &= M_t(2-k) + DVA_t(2-k;k) - CVA_t(2-k\,|\,k) \end{aligned}$$

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where

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Similarly (due to the MCP and permutation):

$$V_t(k) \overset{(2)}{=} M_t(k) - BVA_t(k;2-k) \overset{(!)}{=} -V_t(2-k) \tag{2}$$

for $\mathbb{Q}$-almost all $0 \le t < \tau^*$. Hence, both parties agree. More risky parties pay less risky parties in order to trade with them.

Valuation of defaultable Claims V

Actually we have seen more: namely the following fact which completely ignores the construction/definition of DVA:

Observation

Assume that the MCP holds, and suppose that both parties include a possible future default of their respective counterpart. Then

$$V_t(k) := M_t(k) - \big(CVA_t(k\,|\,2-k) - CVA_t(2-k\,|\,k)\big) = -V_t(2-k)$$

for all $k \in \{0, 2\}$ *and* $\mathbb{Q}$*-almost all* $0 \leq t < \tau^*$*, implying that the MCP can be transferred to the vulnerable cash flow* V*.*

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Notice that also at $t = \tau^*$ the values $M_t(k), CVA_t(k \,|\, 2-k)$ and $CVA_t(2-k \,|\, k)$ are "well-defined". In general they do not vanish. Yet $V_t(k) = -V_t(2-k)$ is the agreed price between k and $2-k$ before a (possible) first-to-default event happens!

The DVA paradox I

Let us assume that party k is default-free (such as e. g. the Bank of England (hopefully...)). Then $\tau_k = \infty$ and $CVA_t(2-k \,|\, k) = DVA_t(k; 2-k) = 0$ for $\mathbb{Q}$-almost all $0 \leq t < \tau^* = \min\{\tau_{2-k}, T\}$. If however, party $2-k$ converged to its own default before T, $CVA_t(k \,|\, 2-k) \uparrow \ldots$ if $t \longrightarrow \tau_{2-k}$. Consequently,

$$V_t(k) = M_t(k) - CVA_t(k \,|\, 2-k) \downarrow \ldots \text{ if } t \longrightarrow \tau_{2-k} \, ,$$

implying that the default-free party k would be strongly exposed to an increase of $CVA_t(k \,|\, 2-k)$ - transferred from the risky party $2-k$ to the solvent party k.

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$$V_t(2-k) = -V_t(k) = M_t(2-k) + DVA_t(2-k; k) \uparrow \ldots \text{ if } t \longrightarrow \tau_{2-k}\,!$$

The DVA paradox II

Whenever an entity's credit worsens, it receives a subsidy from its counterparties in the form of a DVA positive mark to market which can be monetised by the entity's bond holders only at their own default. Whenever an entity's credit improves instead, it is effectively taxed as its DVA depreciates.

Wealth is thus transferred from the equity holders of successful companies to the bond holders of failing ones, the transfer being mediated by banks acting as financial intermediaries and implementing the traditional CVA/DVA mechanics.

The main CCR building blocks

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$0 < LGD_k := 1 - R_k \leq 1$	k's (random) Loss Given Default
$D(t,u) := D(0,u)/D(0,t)$	discount factor at time t for time $u > t$ (can be random)

CCR free close-out I

So, how is $BVA_t(k; 2-k) = CVA_t(k \,|\, 2-k) - DVA_t(k; 2-k)$ actually determined? More precisely: what role do play ISDA's close-out rules here?

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Another important piece of notation: For any stochastic process X we put $\widetilde{X}_t := D(0,t)X_t$ and obtain the discounted process $\widetilde{X}$ with numéraire $D(0,\cdot)$.

Fix $k \in \{0, 2\}$, and assume that party $2-k$ defaults first, implying that $\emptyset \neq A^-_{2-k} \overset{\checkmark}{=} \{\tau_{2-k} = \tau^*\}$ $\mathbb{Q}$-a.s. Let $\omega \in A^-_{2-k}$. Suppose that the close-out is settled at $\tau_{2-k}(\omega) \leq T$ (no margin period of risk) and that no collateral is exchanged between party k and party $2-k$ until $\tau_{2-k}(\omega)$.

CCR free close-out II

Let $\omega \in A^{-}_{2-k}$. A CCR free close-out (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

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Hence, since $A^{-}_{2-k} = \{\tau_{2-k} = \tau^*\}$ $\mathbb{Q}$-a.s. the already defined values $V_k(t)$ $(0 \leq t < \tau^*)$ could be extended in the following way:

$$\begin{aligned} V_k(\tau^*) &:= R_{2-k}(M_k(\tau^*))^+ - (-M_k(\tau^*))^+ \\ &= M_k(\tau^*) - LGD_{2-k}(M_k(\tau^*))^+ \end{aligned} \tag{3}$$

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\begin{aligned}
V_k(\tau^*) &:= R_{2-k}(M_k(\tau^*))^+ - (-M_k(\tau^*))^+ \\
&= M_k(\tau^*) - LGD_{2-k}(M_k(\tau^*))^+ \qquad (3) \\
&\neq M_k(\tau^*) + LGD_k(M_{2-k}(\tau^*))^+ \overset{(MCP)}{=} -V_{2-k}(\tau^*)\,.
\end{aligned}
$$

An axiomatic approach to CVA I

Put $\Delta_k := V_k - M_k$. Then

$$\Delta_k(\tau^*) \stackrel{(3)}{=} -LGD_{2-k}(M_k(\tau^*))^+ \,. \qquad (4)$$

Let us further *assume* that also representation (2) can be extended to τ^*:

$$\Delta_k(\tau^*) \equiv \Delta_{\tau^*}(k) = -BVA_{\tau^*}(k; 2-k) \qquad (5)$$

and that $\mathbb{1}_{A^-_{2-k}} \widetilde{BVA}_\bullet(k; 2-k)$ is a càdlàg UI-$\mathbf{G}$-martingale.

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Let $0 \leq t < \tau^*$ $\mathbb{Q}$-a.s. Equation (2) and an application of the Optional Sampling Theorem imply

$$\begin{aligned} \mathbb{1}_{A^-_{2-k}} \widetilde{\Delta}_t(k) &\overset{(2)}{=} -\mathbb{1}_{A^-_{2-k}} \widetilde{BVA}_t(k; 2-k) \overset{(5)}{=} \mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{1}_{A^-_{2-k}} \widetilde{\Delta}_{\tau^*}(k) \right] \\ &\overset{(4)}{=} -\mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{1}_{A^-_{2-k}} LGD_{2-k}(\widetilde{M}_k(\tau^*))^+ \right] . \qquad (6) \end{aligned}$$

An axiomatic approach to CVA II

Observe that

$$\widetilde{\Delta}_t(k) \overset{(2)}{=} -\widetilde{BVA}_t(k;2-k) = \widetilde{BVA}_t(2-k;k) \overset{(2)}{=} -\widetilde{\Delta}_t(2-k)\,.$$

Consequently (since equation (7) holds for all $k \in \{0,2\}$), we obtain

$$\begin{aligned}
\widetilde{\Delta}_t(k) &= \mathbb{1}_{A_k^-}\widetilde{\Delta}_t(k) + \mathbb{1}_{A_{2-k}^-}\widetilde{\Delta}_t(k) + \mathbb{1}_N\widetilde{\Delta}_t(k) \\
&= -\mathbb{1}_{A_k^-}\widetilde{\Delta}_t(2-k) + \mathbb{1}_{A_{2-k}^-}\widetilde{\Delta}_t(k) + \mathbb{1}_N\widetilde{\Delta}_t(k) \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-}LGD_k(\widetilde{M}_{2-k}(\tau^*))^+\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{A_{2-k}^-}LGD_{2-k}(\widetilde{M}_k(\tau^*))^+\right] \\
&+ \mathbb{1}_N\widetilde{\Delta}_t(k)\,.
\end{aligned}$$

Next, let us assume that $\widetilde{\Delta}_t(k) = 0$ on N (a very “reasonable” assumption since neither party k nor party $2-k$ will default before T). Really?

An axiomatic approach to CVA III

Standing Assumption (B-Zero)

$$\mathbb{1}_N \widetilde{BVA}_t(k; 2-k) = 0$$

for all $k \in \{0, 2\}$ *and for* $\mathbb{Q}$*-almost all* $0 < t < \tau^*$*.*

Vulnerable cash flows I

Let $0 \leq t < \tau^* \ \mathbb{Q}$-a.s. Keeping the assumption (B-Zero) in mind let us revisit Brigo-Capponi's construction of the following vulnerable cash flow (which actually is an "existence result" – cf. [4]):

$$\widehat{\Pi}_k^{(t,T]} := \mathbb{1}_N \Pi_k^{(t,T]} + \mathbb{1}_{A_{2-k}^-} \Pi_k^{(2-k)}(t) + \mathbb{1}_{A_k^-} \Pi_k^{(k)}(t)\,, \qquad (7)$$

where – due to the CCR free close-out rule of ISDA(!) and an application of the MCP – the 2×2 random matrix $(\Pi_k^{(l)}(t))_{l,k \in \{0,2\}}$ is given by

$$\Pi_k^{(l)}(t) \quad := \quad \Pi_k^{(t,\tau^*]} + (-1)^{\frac{k+l}{2}} D(t,\tau^*)\Big(LGD_l\big(M_l(\tau^*)\big)^- + M_l(\tau^*)\Big)$$

for all $l \in \{k, 2-k\}$.

Vulnerable cash flows II

Lemma (Representation of $\widehat{\Pi}_k^{(t,T]}$)

Let $0 \leq t < \tau^* \; \mathbb{Q}$*-a.s. Put* $\Delta_k^{(t,T]} := \widehat{\Pi}_k^{(t,T]} - \Pi_k^{(t,T]}$*. Then*

$$\begin{aligned}
\Delta_k^{(t,T]} &= -\mathbb{1}_{A_{2-k}^-} LGD_{2-k} D(t,\tau^*)\big(M_{2-k}(\tau^*)\big)^- \\
&+ \mathbb{1}_{A_k^-} LGD_k D(t,\tau^*)\big(M_k(\tau^*)\big)^- \\
&+ (1-\mathbb{1}_N) D(t,\tau^*)\left(M_k(\tau^*) - \Pi_k^{(\tau^*,T]}\right).
\end{aligned}$$

Vulnerable cash flows II

Lemma (Representation of $\widehat{\Pi}_k^{(t,T]}$)

Let $0 \leq t < \tau^*$ $\mathbb{Q}$*-a.s. Put* $\Delta_k^{(t,T]} := \widehat{\Pi}_k^{(t,T]} - \Pi_k^{(t,T]}$*. Then*

$$\begin{aligned}
\Delta_k^{(t,T]} &= -\mathbb{1}_{A^-_{2-k}} LGD_{2-k} D(t,\tau^*) \big(M_{2-k}(\tau^*)\big)^- \\
&+ \mathbb{1}_{A^-_k} LGD_k D(t,\tau^*) \big(M_k(\tau^*)\big)^- \\
&+ (1 - \mathbb{1}_N) D(t,\tau^*) \Big(M_k(\tau^*) - \Pi_k^{(\tau^*,T]}\Big) .
\end{aligned}$$

Hence, $\widehat{\Pi}_k^{(t,T]} \overset{(MCP)}{=} -\widehat{\Pi}_{2-k}^{(t,T]}$ *and*

$$\mathbb{E}^{\mathbb{Q}}\Big[(1 - \mathbb{1}_N) D(t,\tau^*) \Big(M_k(\tau^*) - \Pi_k^{(\tau^*,T]}\Big) \Big| \mathcal{G}_{\tau^*}\Big] \overset{(!)}{=} 0$$

$(\mathbb{1}_N,\, D(t,\tau^*)$ *are* $\mathcal{G}_{\tau^*}$*-measurable and* $M_k(\tau^*) = \mathbb{E}^{\mathbb{Q}}_{\tau^*}\big[\Pi_k^{(\tau^*,T]}\big]$*).*

Corollary (Brigo-Capponi (2009))

Let $0 \leq t < \tau^*$ $\mathbb{Q}$*-a.s. Then*

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] &\overset{(!)}{=} M_t(k) \\ &- \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{2-k}^-} LGD_{2-k} D(t,\tau^*)\left(M_k(\tau^*)\right)^+ \middle|\mathcal{G}_t\right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} LGD_k D(t,\tau^*)\left(M_{2-k}(\tau^*)\right)^+ \middle|\mathcal{G}_t\right]. \end{aligned}$$

Corollary (Brigo-Capponi (2009))

Let $0 \leq t < \tau^* \, \mathbb{Q}$*-a.s. Then*

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] &\overset{(!)}{=} M_t(k) \\ &- \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{2-k}^-} LGD_{2-k} D(t,\tau^*) \left(M_k(\tau^*)\right)^+ \middle| \mathcal{G}_t\right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} LGD_k D(t,\tau^*) \left(M_{2-k}(\tau^*)\right)^+ \middle| \mathcal{G}_t\right]. \end{aligned}$$

Proof.

We now only have to recall the "tower property" of conditional expectation:

$$\mathbb{E}^{\mathbb{Q}}[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t] \overset{(!)}{=} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_{\tau^*}\right] \middle| \mathcal{G}_t\right].$$

□

FTDCVA, FTDDVA and FTDBVA I

Let $0 \le t < \tau^* \; \mathbb{Q}$-a.s. Put

$$\begin{aligned}
\text{FTDCVA}_t(k \,|\, 2-k) &:= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{A^-_{2-k}} LGD_{2-k} D(t, \tau_{2-k}) (M_k(\tau_{2-k}))^+\right] \\
&\overset{\checkmark}{\leq} \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{A^-_{2-k}} LGD_{2-k} D(t, \tau^*) (M_k(\tau^*))^+\right], \\
\text{FTDDVA}_t(k; 2-k) &:= \text{FTDCVA}_t(2-k \,|\, k), \\
\text{FTDBVA}_k(t; T) &:= \text{FTDCVA}_t(k \,|\, 2-k) - \text{FTDDVA}_t(k; 2-k)\,.
\end{aligned}$$

FTDCVA, FTDDVA and FTDBVA II

Definition

Let $k = 0$ or $k = 2$ and $0 \leq t < \tau^*$ $\mathbb{Q}$-a.s.

(i) The positive $\mathcal{G}_t$-measurable random variable $\text{FTDCVA}_t(k \,|\, 2-k)$ is called First-to-Default Credit Valuation Adjustment at t.

(ii) The positive $\mathcal{G}_t$-measurable random variable $\text{FTDDVA}_t(k; 2-k) := \text{FTDCVA}_t(2-k \,|\, k)$ is called First-to-Default Debit Valuation Adjustment at t.

(iii) The real $\mathcal{G}_t$-measurable random variable $\text{FTDBVA}_t(k; 2-k) := \text{FTDCVA}_t(k \,|\, 2-k) - \text{FTDDVA}_t(k; 2-k)$ is called First-to-Default Bilateral Valuation Adjustment at t.

Brigo-Capponi reformulated

Theorem (Brigo-Capponi (2009))

Assume that the MCP holds and that the underlying financial market model is arbitrage-free. Let $0 \leq t < \tau^*$ $\mathbb{Q}$*-a.s. Let* $M_t(k)$ *denote the mark-to-market value of the portfolio to party* k *in case both,* 0 *and* 2 *do not default until* T*. If both parties apply the CCR free close-out rule it follows that*

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$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - \mathit{FTDBVA}_t(k; 2-k)$$

for all $k \in \{0, 2\}$*.*

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for all $k \in \{0, 2\}$*.*

We even may formulate the following

Bilateral CCR risk premium vs FTDBVA

Theorem

Let $k \in \{0, 2\}$. Assume that the MCP holds and that the underlying financial market model is arbitrage-free. Let $0 \leq t < \tau^* \; \mathbb{Q}$-a.s. Let $V_t(k) := M_t(k) - B_t(k; 2-k)$, where

- *$M_t(k)$ denotes the mark-to-market value of the portfolio to party k in case both, 0 and 2 do not default until T,*
- *$B_t(k; 2-k) = -B_t(2-k; k)$ is $\mathbf{G}$-adapted,*
- *$V_{\tau^*}(k) = M_{\tau^*}(k) - B_{\tau^*}(k; 2-k)$,*
- *$\mathbb{1}_{A_{2-k}^-} \widetilde{B}_\bullet(k; 2-k)$ is a càdlàg and uniformly integrable $\mathbf{G}$-martingale which satisfies condition (B-Zero).*

If both parties 0 and 2 apply the CCR free close-out rule, then $B_t(k; 2-k) =$ FTDBVA$_t(k; 2-k)$ and $V_t(k) = \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\big]$.

Excerpt from Basel III (ACVA, Para 98)

A. Banks with IMM approval and Specific Interest Rate Risk VaR model[34] approval for bonds: Advanced CVA risk capital charge

98. Banks with IMM approval for counterparty credit risk and approval to use the market risk internal models approach for the specific interest-rate risk of bonds must calculate this additional capital charge by modelling the impact of changes in the counterparties' credit spreads on the CVAs of all OTC derivative counterparties, together with eligible CVA hedges according to new paragraphs 102 and 103, using the bank's VaR model for bonds. This VaR model is restricted to changes in the counterparties' credit spreads and does not model the sensitivity of CVA to changes in other market factors, such as changes in the value of the reference asset, commodity, currency or interest rate of a derivative. Regardless of the accounting valuation method a bank uses for determining CVA, the CVA capital charge calculation must be based on the following formula for the CVA of each counterparty:

$$CVA = (LGD_{MKT}) \cdot \sum_{i=1}^{T} Max\left(0; exp\left(-\frac{s_{i-1} \cdot t_{i-1}}{LGD_{MKT}}\right) - exp\left(-\frac{s_i \cdot t_i}{LGD_{MKT}}\right)\right) \cdot \left(\frac{EE_{i-1} \cdot D_{i-1} + EE_i \cdot D_i}{2}\right)$$

Where

- t_i is the time of the i-th revaluation time bucket, starting from t_0=0.
- t_T is the longest contractual maturity across the netting sets with the counterparty.

UCVA$_t(k \,|\, 2-k)$ as a special case of FTDCVA$_t(k \,|\, 2-k)$

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Special Case (A single default only $\rightsquigarrow$ **Basel III**)

Fix $k \in \{0, 2\}$. *Assume that in addition* $\tau_k = +\infty$ (*i. e., no default of party* k). *Then* $A^-_{2-k} = \{\tau_{2-k} \leq T\}$ ($\mathbb{Q}$*-a.s*)*. and* $A^-_k = \emptyset$*. Consequently, FTDDVA*$_t(k; 2-k) = 0$*,*

$$\mathbb{E}^{\mathbb{Q}}\big[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\big] = M_t(k) - \mathit{FTDCVA}_t(k \,|\, 2-k)\,,$$

UCVA$_t(k\,|\,2-k)$ as a special case of FTDCVA$_t(k\,|\,2-k)$

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$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] = M_t(k) - \textit{FTDCVA}_t(k\,|\,2-k)\,,$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]}|\mathcal{G}_t\right] = M_t(2-k) + \textit{FTDDVA}_t(2-k;k).$$

$\text{UCVA}_t(k\,|\,2-k)$ as a special case of $\text{FTDCVA}_t(k\,|\,2-k)$

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$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - \mathit{FTDCVA}_t(k\,|\,2-k)\,,$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]} | \mathcal{G}_t\right] = M_t(2-k) + \mathit{FTDDVA}_t(2-k; k).$$

Hence, if party k *were the investor, and if* $\tau_k = +\infty$ *the Unilateral CVA UCVA*$_t(k, 2-k) :=$ *FTDCVA*$_t(k\,|\,2-k)$ *would have to be paid by party* $2-k$ *to the default free party* k *at* t *to cover a potential default of party* $2-k$ *after* t.

Structure of FTDCVA$_t(k \mid 2-k)$

Although we write "FTDCVA$_t(k \mid 2-k)$" it always should be kept in mind that we actually are working with a very complex object, namely:

$$\boxed{\text{FTDCVA}_k(t, T, LGD_{2-k}, \tau_k, \tau_{2-k}, D(t, \tau_{2-k}), M_k(\tau_{2-k}))} \quad !$$

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In the following – similarly to Basel III – we consider the case $t = 0$ "only". Why?

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In the following – similarly to Basel III – we consider the case $t = 0$ "only". Why? The case $0 < t < \tau^*$ requires an in depth analysis of the conditional joint default process

$$\Big(\mathbb{Q}\big(\tau_k \leq T \text{ and } \tau_{2-k} \leq \tau_k \big| \mathcal{G}_t\big)\Big)_{t \geq 0}.$$

Structure of $\text{FTDCVA}_t(k \,|\, 2-k)$

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To cover dynamically changing stochastic dependence between all embedded risk factors, a truly dynamic copula model has to be constructed ($\rightsquigarrow$ Bielecki, Crépey, Jeanblanc et al).

UCVA and Basel III - Part I

Firstly we list a very restrictive case of a possible calculation of UCVA, encoded in the much too simple "CVA $=$ PD $*$ LGD $*$ EE" formula which however seems to be used often in financial institutes.

Proposition (Rough Approximation – Part I)

Let $k \in \{0, 2\}$. Assume that

(i) *party k will not default until T: $\tau_k := +\infty$;*

(ii) *LGD_{2-k} is constant and non-random;*

(iii) *$(\widetilde{M}_k(\tau_{2-k}))^+$ and τ_{2-k} are independent under $\mathbb{Q}$ (i. e., WWR or RWR is ignored completely).*

Then

$$UCVA_0(k \,|\, 2-k) = \mathbb{Q}(\tau_{2-k} \leq T) \cdot LGD_{2-k} \cdot \mathbb{E}^{\mathbb{Q}}\big[(\widetilde{M}_k(\tau_{2-k}))^+\big].$$

UCVA and Basel III - Part II

Suppose there exists a further random variable $\mathfrak{M}$ (a "market risk factor") so that $M_k(\tau_{2-k})$ is a function of $\mathfrak{M}$ as well, $M_k(\tau_{2-k}, \mathfrak{M})$ say.

Proposition (Rough Approximation – Part II)

Assume that

(i) *Party k will not default until T: $\tau_k := +\infty$;*

(ii) *LGD_{2-k} is constant and non-random;*

(iii) *For all t $D(0,t)$ does not depend on $\mathfrak{M}$;*

(iv) *$\mathfrak{M}$ and τ_{2-k} are independent under $\mathbb{Q}$.*

Then

$$UCVA_k(0 \mid T) = LGD_{2-k} \int_0^T D(0,t)\mathbb{E}^{\mathbb{Q}}[(M_k(t,\mathfrak{M}))^+]\, dF^{\mathbb{Q}}_{\tau_{2-k}}(t),$$

where $F^{\mathbb{Q}}_{\tau_{2-k}}(t) := \mathbb{Q}(\tau_{2-k} \leq t)$ for all $t \in \mathbb{R}$ (unconditional df).

Proof.

Put $\Phi(t,m) := \mathbb{1}_{[0,T]}(t) \cdot \psi(t,m)$, where $(t,m)^\top \in \mathbb{R}^+ \times \mathbb{R}$ and $\psi(t,m) := D(0,t) \cdot (M_k(t,m))^+$. Let $F^{\mathbb{Q}}_{(\tau_{2-k},\mathfrak{M})}$ denote the *bivariate* df of the random vector $(\tau_{2-k},\mathfrak{M})^\top$ w.r.t. $\mathbb{Q}$. Then

$$
\begin{aligned}
\text{UCVA}_0(k\,|\,2-k) &\overset{(i),(ii)}{=} LGD_{2-k}\mathbb{E}^{\mathbb{Q}}[\Phi(\tau_{2-k},\mathfrak{M})] \\
&= LGD_{2-k}\int_{\mathbb{R}^+\times\mathbb{R}} \Phi(t,m)dF^{\mathbb{Q}}_{(\tau_{2-k},\mathfrak{M})}(t,m) \\
&\overset{(iv),\text{Fubini}}{=} LGD_{2-k}\int_{[0,T]}\Big(\int_{\mathbb{R}}\psi(t,m)dF^{\mathbb{Q}}_{\mathfrak{M}}(m)\Big)dF^{\mathbb{Q}}_{\tau_{2-k}}(t) \\
&\overset{(iii)}{=} LGD_{2-k}\int_0^T D(0,t)\mathbb{E}^{\mathbb{Q}}[(M_k(t,\mathfrak{M}))^+]dF^{\mathbb{Q}}_{\tau_{2-k}}(t).
\end{aligned}
$$

□

EE, Wrong-Way Risk and Right-Way Risk

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$\mathrm{EE}_k^{(\mathfrak{M})}(t) := \mathbb{E}^{\mathbb{Q}}[(M_k(t,\mathfrak{M}))^+]$ is known as party k's Expected Exposure at t. In general it can be identified by MC simulation only.

EE, Wrong-Way Risk and Right-Way Risk

$\mathrm{EE}_k^{(\mathfrak{M})}(t) := \mathbb{E}^{\mathbb{Q}}[(M_k(t,\mathfrak{M}))^+]$ is known as party k's Expected Exposure at t. In general it can be identified by MC simulation only.

The situation where $\mathbb{Q}(\tau_{2-k} \leq t)$ is positively dependent on $\mathrm{EE}_k^{(\mathfrak{M})}(t)$, is referred to as Wrong-Way Risk (WWR). In the case of WWR, there is a tendency for party $2-k$ to default when party k's exposure to party $2-k$ is relatively high. The situation where $\mathbb{Q}(\tau_{2-k} \leq t)$ is negatively dependent on $\mathrm{EE}_k^{(\mathfrak{M})}(t)$ is referred to as Right-Way Risk (RWR). In the case of RWR, there is a tendency for party $2-k$ to default when party k's exposure to party $2-k$ is relatively low (cf. [5], [6]).

Is the ISDA formula (Para 98) of Basel III true?

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Technical Remark

Regarding the calculation of $UCVA_0(k \,|\, 2-k)$ *in Basel III (para 98), observe that the integral in the above Proposition in fact is a Lebesgue-Stieltjes integral. Hence, if* $t \mapsto EE_k^{(\mathfrak{M})}(t)$ *were not continuous (in time) and if it oscillated too strongly, that integral would not necessarily be a Riemann-Stieltjes integral, implying that we seemingly cannot simply approximate it numerically through a Riemann-Stieltjes sum of the type*

$$\begin{aligned} UCVA_0(k \,|\, 2-k) &\approx \sum_{i=1}^{n} D(0, t_i^*) \cdot EE_k^{(\mathfrak{M})}(t_i^*) \cdot \left(F_{\tau_{2-k}}^{\mathbb{Q}}(t_i) - F_{\tau_{2-k}}^{\mathbb{Q}}(t_{i-1})\right) \\ &= \sum_{i=1}^{n} D(0, t_i^*) \cdot EE_k^{(\mathfrak{M})}(t_i^*) \cdot \mathbb{Q}(t_{i-1} < \tau_{2-k} \leq t_i), \quad (8) \end{aligned}$$

Basel III UCVA slightly modified

where $0 = t_0 < \ldots < t_n = T$ and $2t_i^* := t_{i-1} + t_i$. However:

Corollary

Assume that

(i) *The assumptions (i), (ii) and (iv) of the previous Proposition are satisfied;*

(ii) *For all $i = 1, \ldots, n$, for all $t \in [t_{i-1}, t_i]$,* $\mathbb{Q}(\tau_{2-k} > t) = \exp\big(-\lambda_{2-k}^{(i)}\, t\big)$, *where $\lambda_{2-k}^{(i)} > 0$ is a constant;*

(iii) *For all $i = 1, \ldots, n$, for all $t \in [t_{i-1}, t_i]$, $r(t) \equiv r_i$ is constant;*

(iv) *LGD_{2-k} is calibrated from a CDS curve with constant CDS spread $s_{2-k}^{(i)}$ on each $[t_{i-1}, t_i]$.*

Then $s_{2-k}^{(i)} = \lambda_{2-k}^{(i)} \cdot LGD_{2-k}$ ("Credit Triangle"), and

$$UCVA_0(k \,|\, 2-k) = \sum_{i=1}^{n} s_{2-k}^{(i)} \int_{t_{i-1}}^{t_i} e^{-r_i\, t} EE_k^{(\mathfrak{M})}(t) \exp\left(-\frac{s_{2-k}^{(i)} t}{LGD_{2-k}}\right) dt.$$

CVA risk in Basel III (Para 99)

Assuming both, the approximation (8) of Basel III and the "spread representation"

$$\mathbb{Q}(t_{i-1}^* < \tau_{2-k} \leq t_i^*) = e\big(s_{2-k}^{(i-1)}, t_{i-1}^*\big) - e\big(s_{2-k}^{(i)}, t_i^*\big),$$

where $e(s,t) := \exp(-s \cdot t / LGD_{2-k})$, a Taylor series approximation of 2^{nd} order leads to the so called "CVA risk" of Basel III, i. e., to a delta/gamma approximation for $\text{UCVA}_0(k \,|\, 2-k)$, viewed as a function $f(s_{2-k})$ of the n-dimensional spread vector $s_{2-k} \equiv (s_{2-k}^{(1)}, \ldots s_{2-k}^{(n)})^\top$ only:

$$f(s_{2-k} + h) - f(s_{2-k}) \overset{(\|h\| \text{ small })}{\approx}$$

$$\sum_{i=1}^{n} D(0, t_i^*) \mathsf{EE}_k^{(\mathfrak{M})}(t_i^*)\, h_i \big(t_i^* e\big(s_{2-k}^{(i)}, t_i^*\big) - t_{i-1}^* e\big(s_{2-k}^{(i-1)}, t_{i-1}^*\big)\big) \;\; +$$

$$\frac{1}{2\,LGD_{2-k}} \sum_{i=1}^{n} D(0, t_i^*) \mathsf{EE}_k^{(\mathfrak{M})}(t_i^*)\, h_i^2 \big(t_{i-1}^{*2} e\big(s_{2-k}^{(i-1)}, t_{i-1}^*\big) - t_i^{*2} e\big(s_{2-k}^{(i)}, t_i^*\big)\big)$$

CVA risk in Basel III: Flaws I

An analysis of “CVA volatility risk” and its capitalisation should particularly treat the following serious flaws:

(i) CVA risk (and hedges) extend far beyond the risk of credit spread changes. It includes all risk factors that drive the underlying counterparty exposures as well as dependent interactions between counterparty exposures and the credit spreads of the counterparties (and their underyings). By solely focusing on credit spreads, the Basel III UCVA VaR and stressed VaR measures in its advanced approach for determining a CVA risk charge do not reflect the real risks that drive the P&L and earnings of institutes. Moreover, banks typically hedge these non-credit-spread risk factors. The Basel III capital calculation does not include these hedges.

CVA risk in Basel III: Flaws II

(ii) The non-negligible and non-trivial problem of a more realistic inclusion of WWR should be analysed deeply. In particular, the “alpha” multiplier $1.2 \leq \alpha$ should be revisited, and any unrealistic independence assumption should be strongly avoided.

(iii) Credit and market risks in UCVA are not different from the same risks, embedded in many other trading positions such as corporate bonds, CDSs, or equity derivatives. CVA risk can be seen as just another source of market risk. Consequently, it should be managed within the trading book. Basel III requires that the CVA risk charge is calculated on a stand alone basis, separated from the trading book. This seems to be an artificial segregation. A suitable approach would be to include UCVA and all of its hedges into the trading book capital calculation.

CVA risk in Basel III: Flaws III

(iv) Basel III considers unilateral CVA only. More precisely, the regulatory calculation of the ACVA is based on $UCVA_0$ – as opposed to the calculations of CVA in FAS 157 respectively IAS 39! Latter explicitly include the $(U)DVA_0$. Hence, there exists a non-trivial mismatch between regulation and accounting! Moreover, as we have seen a thorough and appropriate treatment of a market price of (bilateral) CCR leads to $FTDBVA_0$ and not to $UCVA_0$. Consequently, further research is necessary. There is work in progress such as e.g. the running "Fundamental Review of the Trading Book" or running projects in the RTF subgroup of the BCBS – hopefully leading to necessary improvements of Basel III.

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Thank you for your attention!

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Are there any questions, comments or remarks?