Operator ideals and approximation properties of Banach spaces

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Why operator ideals? I

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$$\mathcal{L}^2(E,F) \cong \mathcal{L}(E,F') \overset{(!)}{\cong} G'$$

where $E, F \in BAN$ and $G := E\widetilde{\otimes}_\pi F$.

Why operator ideals? II

Let G be an arbitrary locally compact group with left Haar measure μ, and let $1 \leq p < \infty$. Recall that the group algebra $(L^1(G), *)$ is a Banach algebra for the convolution product, defined by

$$f * g(t) := \int_G f(s)g(s^{-1}t)d\mu(s) = \int_G f(ts)g(s^{-1})d\mu(s)\,.$$

The Banach space $L^p(G)$ is a Banach left $L^1(G)$-module in a canonical way (by using convolution again).

Why operator ideals? II

Theorem (B. E. Johnson, 1972)

Suppose that G is an amenable locally compact group and $1 < p < \infty$. Then $L^p(G)$ is an injective Banach left $L^1(G)$-module.

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Multi-norm approach (H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, 2012)

Let E be a normed space. Suppose that $p \geq 1$. Then the p-multi-norm on $\{E^n : n \in \mathbb{N}\}$ induces the norm on $c_0 \otimes E$, given by the isometric embedding

$$c_0 \otimes E \overset{1}{\hookrightarrow} \mathcal{P}_p(E', c_0)$$

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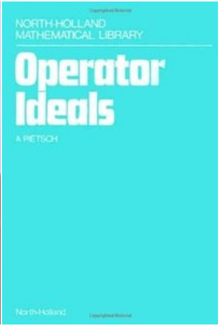
Corollary (H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, (2012))

Suppose that G is a locally compact group. If $L^p(G)$ is an injective Banach left $L^1(G)$-module for some (and hence all) $1 < p < \infty$, G is amenable.

Operator ideals in Jena: Albrecht Pietsch



$$s_n(RST) \le \|R\|\, s_n(S)\|T\|,\ T \in \mathcal{L}(E_1, E), S \in \mathcal{L}(E, F), R \in \mathcal{L}(F, F_1)$$

$$\prod_{k=1}^{n} |\lambda_k(T)| \le n^{n/2} \prod_{k=1}^{n} a_k(T)$$

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(I1) $Id_{\mathbb{K}} \in \mathcal{A}(\mathbb{K},\mathbb{K})$;

(I2) If E_0 and F_0 are Banach spaces, then $RST \in \mathcal{A}(E_0,F_0)$, whenever $T \in \mathcal{L}(E_0,E), S \in \mathcal{A}(E,F)$ and $R \in \mathcal{L}(F,F_0)$.

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(N2) $\|RST\|_{\mathcal{A}} \leq \|R\| \, \|S\|_{\mathcal{A}} \, \|T\|$, whenever E_0 and F_0 are Banach spaces and $T \in \mathcal{L}(E_0, E)$, $S \in \mathcal{A}(E, F)$ and $R \in \mathcal{L}(F, F_0)$,

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(N3) $(\mathcal{A}(E,F), \|\cdot\|_{\mathcal{A}})$ is a Banach space,

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Operator ideals: definition III

Definition (ctd.)

Further, if for each pair of Banach spaces (E, F), $\mathcal{A}(E, F)$ is supplied with a p-norm $\|\cdot\|_{\mathcal{A}}$, with $0 < p \leq 1$, satisfying

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Banach ideals: important examples I

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Banach ideals: important examples I

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- $(\mathcal{K}, \|\cdot\|)$: the Banach ideal of all compact linear operators;
- $(\mathcal{W}, \|\cdot\|)$: the Banach ideal consisting of all weakly compact operators (coinciding with the class of all those bounded linear operators between Banach spaces which factor through a reflexive Banach space).

Banach ideals: important examples II

- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$: the smallest *Banach* ideal consisting of the class of all nuclear operators between Banach spaces:

  $T \in \mathcal{N}(E,F)$ iff there exist sequences $(a_n)_{n\in\mathbb{N}} \subseteq E'$ and $(y_n)_{n\in\mathbb{N}} \subseteq F$ such that $\sum_{n=1}^{\infty} \|a_n\| \|y_n\| < \infty$ and $T = \sum_{n=1}^{\infty} \langle \cdot, a_n \rangle y_n = \sum_{n=1}^{\infty} a_n \underline{\otimes} y_n$, implying that $\|T\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|a_n\| \|y_n\| \,:\, T = \ldots \right\}$ is well-defined.

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- $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$: the Banach ideal of all integral operators:

  $T \in \mathcal{I}(E,F)$ iff there exists a constant $c \geq 0$ such that for all finite rank operators $L \in \mathcal{F}(F,E)$

  $$|\mathrm{tr}(TL)| \leq c \, \|L\| \; .$$

  $\|T\|_{\mathcal{I}} := \inf\{c : c \text{ satisfies } \ldots\}$ is well-defined for any integral operator T.

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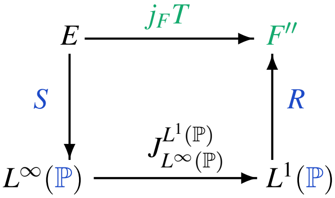
Banach ideals: important examples III

Theorem (Grothendieck – 1956)

$T \in \mathcal{I}(E,F)$ if and only if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and operators $S \in \mathcal{L}(E, L^\infty(\mathbb{P}))$, $R \in \mathcal{L}\big(L^1(\mathbb{P}), F''\big)$, such that the following diagram commutes



and $\|T\|_{\mathcal{I}} = \inf\{\|R\|\,\|S\| \,:\, \dots\}$, where the infimum is taken over all possible $\mathbb{P}$'s, R's, and S's.

Banach ideals: important examples III

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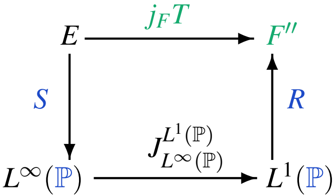

and $\|T\|_{\mathcal{I}} = \inf\{\|R\|\,\|S\| \ : \ldots\}$, where the infimum is taken over all possible $\mathbb{P}$'s, R's, and S's.

Main Idea of a Proof (dualisation of tensor products !)

Put $\Omega := B_{E'} \times B_{F''}$. Then $E \otimes_\varepsilon F' \overset{1}{\hookrightarrow} L^\infty(\Omega, \mathbb{P}) \otimes_\varepsilon (L^1(\Omega, \mathbb{P}))'$...

Banach ideals: important examples IV

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- Fix $1 \leq p < \infty$, and consider $\left(\mathcal{P}_p, \|\cdot\|_{\mathcal{P}_p}\right)$, the Banach ideal of all absolutely p-summing operators; i. e., the class of all operators between Banach spaces which map weakly p-summable sequences in E to strongly p-summable sequences in F.

Banach ideals: important examples IV

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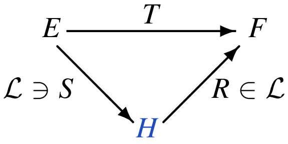
$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{\frac{1}{p}} \leq c \sup\Big\{\Big(\sum_{k=1}^{n} |\langle x_k, a\rangle|^p\Big)^{\frac{1}{p}} : a \in B_{E'}\Big\},$$

implying that $\|T\|_{\mathcal{P}_p} := \inf\{c : c \text{ satisfies } \ldots\}$ is well-defined for any absolutely p-summing operator T.

Banach ideals: important examples V

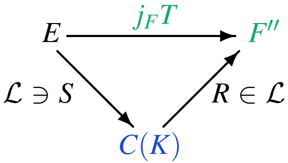
- $(\mathcal{L}_2, \|\cdot\|_{\mathcal{L}_2})$: the Banach ideal consisting of all operators between Banach spaces which factor through a Hilbert space:



$\|T\|_{\mathcal{L}_2} := \inf\{\|R\|\,\|S\| \;:\; T = RS \ldots\}$, where the infimum is taken over all possible Hilbert spaces H and factorising R's and S's.

Banach ideals: important examples VI

- $(\mathcal{L}_\infty, \|\cdot\|_{\mathcal{L}_\infty})$: the Banach ideal consisting of all operators between Banach spaces which factor through some $C(K)$, where K is a compact set, in the following sense:



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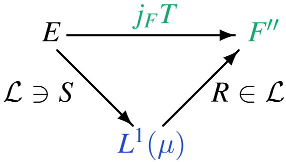
Banach ideals: important examples VII

- $(\mathcal{L}_1, \|\cdot\|_{\mathcal{L}_1})$: the Banach ideal consisting of all operators between Banach spaces which factor through some $L^1(\mu)$, where μ is a Borel-Radon measure, in the following sense:



$\|T\|_{\mathcal{L}_1} := \inf\{\|R\|\,\|S\| \ : T = RS \ldots\}$, where the infimum is taken over all possible spaces $L^1(\mu)$ (μ a Borel-Radon measure) and factorising R's and S's.

In the following, if not differently stated, let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ always be an arbitrary p-Banach ideal and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ always an arbitrary q-Banach ideal, where $0 < p, q \leq 1$.

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Definition

We write

$$(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \subseteq (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$$

iff for all Banach spaces E, F, $\mathcal{A}(E, F) \subseteq \mathcal{B}(E, F)$ ("algebraically") and $\|T\|_{\mathcal{B}} \leq \|T\|_{\mathcal{A}}$ for all $T \in \mathcal{A}(E, F)$.

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If an arbitrary operator ideal is given, one can "derive canonically" further operator ideals, such as e. g. the following very important candidates:

Dual operator ideal and regular hull

- $(\mathcal{A}^{\text{dual}}, \|\cdot\|_{\mathcal{A}^{\text{dual}}})$: the dual p-Banach ideal. $T \in \mathcal{A}^{\text{dual}}(E, F)$ iff $T' \in \mathcal{A}(F', E')$; $\|T\|_{\mathcal{A}^{\text{dual}}} := \|T'\|_{\mathcal{A}}$;

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- $(\mathcal{A}^{\text{reg}}, \|\cdot\|_{\mathcal{A}^{\text{reg}}})$: the regular hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. $T \in \mathcal{A}^{\text{reg}}(E, F)$ iff $j_F T \in \mathcal{A}(E, F'')$, where $j_F : F \overset{1}{\hookrightarrow} F''$ is the canonical injection. $\|T\|_{\mathcal{A}^{\text{reg}}} := \|j_F T\|_{\mathcal{A}}$;

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Recall that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is regular iff $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{reg}}, \|\cdot\|_{\mathcal{A}^{\text{reg}}})$.

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- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}}) \overset{(!)}{\neq} (\mathcal{N}^{\textit{reg}}, \|\cdot\|_{\mathcal{N}^{\textit{reg}}}) = (\mathcal{N}^{\textit{dual}}, \|\cdot\|_{\mathcal{N}^{\textit{dual}}})$;

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Examples

- $\left(\mathcal{N}, \|\cdot\|_{\mathcal{N}}\right) \overset{(!)}{\neq} \left(\mathcal{N}^{\textit{reg}}, \|\cdot\|_{\mathcal{N}^{\textit{reg}}}\right) = \left(\mathcal{N}^{\textit{dual}}, \|\cdot\|_{\mathcal{N}^{\textit{dual}}}\right)$;
- $\left(\mathcal{K}^{\textit{dual}}, \|\cdot\|\right) = \left(\mathcal{K}, \|\cdot\|\right)$;

Dual operator ideal and regular hull

- $(\mathcal{A}^{\text{dual}}, \|\cdot\|_{\mathcal{A}^{\text{dual}}})$: the dual p-Banach ideal. $T \in \mathcal{A}^{\text{dual}}(E,F)$ iff $T' \in \mathcal{A}(F',E')$; $\|T\|_{\mathcal{A}^{\text{dual}}} := \|T'\|_{\mathcal{A}}$;
- $(\mathcal{A}^{\text{reg}}, \|\cdot\|_{\mathcal{A}^{\text{reg}}})$: the regular hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. $T \in \mathcal{A}^{\text{reg}}(E,F)$ iff $j_F T \in \mathcal{A}(E,F'')$, where $j_F : F \overset{1}{\hookrightarrow} F''$ is the canonical injection. $\|T\|_{\mathcal{A}^{\text{reg}}} := \|j_F T\|_{\mathcal{A}}$;

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Examples

- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}}) \overset{(!)}{\neq} (\mathcal{N}^{reg}, \|\cdot\|_{\mathcal{N}^{reg}}) = (\mathcal{N}^{dual}, \|\cdot\|_{\mathcal{N}^{dual}})$;
- $(\mathcal{K}^{dual}, \|\cdot\|) = (\mathcal{K}, \|\cdot\|)$;
- $(\mathcal{P}_1, \|\cdot\|_{\mathcal{P}_1}) = (\mathcal{P}_1^{reg}, \|\cdot\|_{\mathcal{P}_1^{reg}}) \subsetneq (\mathcal{L}_2, \|\cdot\|_{\mathcal{L}_2}) = (\mathcal{L}_2^{reg}, \|\cdot\|_{\mathcal{L}_2^{reg}}) = (\mathcal{L}_2^{dual}, \|\cdot\|_{\mathcal{L}_2^{dual}})$.

Injective and surjective operator ideals

- $\left(\mathcal{A}^{\text{inj}}, \|\cdot\|_{\mathcal{A}^{\text{inj}}}\right)$: the injective hull of $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$. $T \in \mathcal{A}^{\text{inj}}(E,F)$ iff $J_F T \in \mathcal{A}(E, F^\infty)$, where $F^\infty := C(B_{F'})$ and $J_F : F \overset{1}{\hookrightarrow} F^\infty$ is the canonical isometric embedding. $\|T\|_{\mathcal{A}^{\text{inj}}} := \|J_F T\|_{\mathcal{A}}$;

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Examples

- $(\overline{\mathcal{F}}, \|\cdot\|) \overset{(!)}{\subsetneq} (\mathcal{K}, \|\cdot\|) = (\overline{\mathcal{F}}^{inj}, \|\cdot\|) = (\overline{\mathcal{F}}^{sur}, \|\cdot\|)$;

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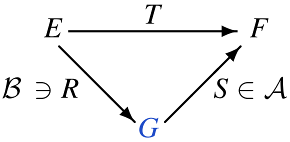
- $(\mathcal{P}_1, \|\cdot\|_{\mathcal{P}_1}) = (\mathcal{I}^{inj}, \|\cdot\|_{\mathcal{I}^{inj}})$ *is not surjective.*

Products of operator ideals I

$T \in \mathcal{L}(E, F)$ is an element of the product ideal $\mathcal{A} \circ \mathcal{B}(E, F)$ iff there exists a Banach space G and operators $R \in \mathcal{B}(E, G)$ and $S \in \mathcal{A}(G, F)$ such that $T = SR$:



$\|T\|_{\mathcal{A}\circ\mathcal{B}} := \inf\{\|S\|_{\mathcal{A}} \cdot \|R\|_{\mathcal{B}} : T = RS \ldots\}$, where the infimum is taken over all possible Banach spaces G, and factorising R's and S's.

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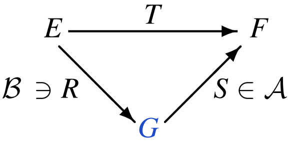

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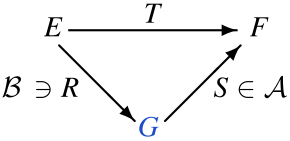

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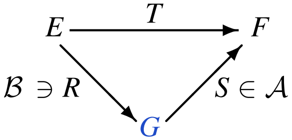

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- $(\mathcal{L}_2 \circ \mathcal{N}, \|\cdot\|_{\mathcal{L}_2\circ\mathcal{N}}) \overset{(!)}{=} (\mathcal{P}_2 \circ \mathcal{P}_2, \|\cdot\|_{\mathcal{P}_2\circ\mathcal{P}_2}) \subsetneq (\mathcal{N}, \|\cdot\|_{\mathcal{N}})$.

Products of operator ideals II

Remark

$(\mathcal{A} \circ \mathcal{B}, \|\cdot\|_{\mathcal{A} \circ \mathcal{B}})$ *is a* r*-Banach ideal, where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *In general,* $\|\cdot\|_{\mathcal{A} \circ \mathcal{B}}$ *is not a norm.*

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Proposition

- $(\mathcal{D}_2, \|\cdot\|_{\mathcal{D}_2}) := (\mathcal{P}_2^{dual} \circ \mathcal{P}_2, \|\cdot\|_{\mathcal{P}_2^{dual}\circ\mathcal{P}_2})$ *is a Banach ideal.*

Products of operator ideals II

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Products of operator ideals II

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- $(\mathcal{L}_2 \circ \mathcal{P}_2^{\mathit{dual}}, \|\cdot\|_{\mathcal{L}_2\circ\mathcal{P}_2^{\mathit{dual}}}) \subseteq (\mathcal{P}_2, \|\cdot\|_{\mathcal{P}_2})$.
- *Let* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \subseteq (\mathcal{D}_2, \|\cdot\|_{\mathcal{D}_2})$. *Then* $(\mathcal{L}_2 \circ \mathcal{A}, \|\cdot\|_{\mathcal{L}_2\circ\mathcal{A}})$ *is not a Banach ideal.*

Products of operator ideals III

Theorem (Grothendieck's inequality in operator form)

Every (bounded linear) operator from l_1 to l_2 is absolutely 1-summing, and

$$\|T\|_{\mathcal{P}_1} \leq K_G \|T\|$$

for some (universal) constant $K_G > 0$.

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respectively,

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Grothendieck and Operations Research

In fact, meanwhile there exist highly non-trivial applications of Grothendieck's inequality in OR, and in the theory of vector valued measures, such as e. g. a construction of an algorithm which combines semidefinite programming with a novel rounding technique, where the latter is based on Grothendieck's inequality! This approach plays a major role in the design of efficient approximation algorithms for dense graph and matrix problems (keyword: "cut-norm" of a real matrix)!

Adjoint operator ideal I

Let us recall that $T \in \mathcal{A}^*(E, F)$ iff there exists a constant $c \geq 0$ such that

$$|\mathrm{tr}\left(TJ_M^E S Q_K^F\right)| \leq c\|S\|_{\mathcal{A}}$$

for all Banach spaces $M \in \mathrm{FIN}(E)$, $K \in \mathrm{COFIN}(F)$, and $S \in \mathcal{L}(F/K, M)$. If we denote

$$\|T\|_{\mathcal{A}^*} := \inf(c),$$

where the infimum is taken over all such constants c, we obtain a Banach ideal $\left(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*}\right)$, the adjoint of $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$.

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Examples

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- $\left(\mathcal{L}^{*}, \|\cdot\|_{\mathcal{L}^{*}}\right) = \left(\mathcal{I}, \|\cdot\|_{\mathcal{I}}\right) = \left(\overline{\mathcal{F}}^{*}, \|\cdot\|_{\overline{\mathcal{F}}^{*}}\right) = \left(\mathcal{K}^{*}, \|\cdot\|_{\mathcal{K}^{*}}\right)$;
- $\left(\mathcal{A}^{\mathit{inj}\,*}, \|\cdot\|_{\mathcal{A}^{\mathit{inj}\,*}}\right) = \left(\left(\mathcal{A}^{*} \circ \mathcal{L}_{\infty}\right)^{reg}, \|\cdot\|_{\left(\mathcal{A}^{*} \circ \mathcal{L}_{\infty}\right)^{reg}}\right)$.

Adjoint operator ideal II

Theorem

Let $\mathcal{B}$ be a maximal Banach ideal such that $\mathcal{B} \subseteq \mathcal{L}_\infty$. Then $\mathcal{B} \circ \mathcal{L}_2$ cannot be a 1-Banach ideal.

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Corollary

Let $\mathcal{B}$ be a maximal Banach ideal. If $\mathcal{B} \circ \mathcal{L}_2$ is a Banach ideal, then $\mathcal{B}^*$ cannot be injective.

Minimal kernel and compact kernel

Of particular importance are the following two product ideal constructions:

- The minimal kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$:

$$(\mathcal{A}^{\text{min}}, \|\cdot\|_{\mathcal{A}^{\text{min}}}) := (\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}, \| \cdot \|_{\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}})$$

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- The compact kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ [Karn-Sinha – 2012]:

$$(\mathcal{A}^{\text{com}}, \|\cdot\|_{\mathcal{A}^{\text{com}}}) := (\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}, \|\cdot\|_{\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}})$$

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Examples

Minimal kernel and compact kernel

Of particular importance are the following two product ideal constructions:

- The minimal kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$:

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- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}}) = (\mathcal{I}^{min}, \|\cdot\|_{\mathcal{I}^{min}})$;
- *Let* $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ *be a Banach ideal. Then* $(\mathcal{C}^* \circ \mathcal{C}^{min}, \| \cdot \|_{\mathcal{C}^* \circ \mathcal{C}^{min}}) \subseteq (\mathcal{N}, \|\cdot\|_{\mathcal{N}})$.

Conjugate operator ideals I

$T \in \mathcal{A}^{\triangle}(E, F)$ iff there exists a constant $c \geq 0$ such that for all finite rank operators $L \in \mathcal{F}(F, E)$

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If we put

$$\|T\|_{\mathcal{A}^{\triangle}} := \inf(c),$$

where the infimum is taken over all such constants c, we obtain a Banach ideal $(\mathcal{A}^{\triangle}, \|\cdot\|_{\mathcal{A}^{\triangle}})$, the conjugate of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$.

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It is trivial to see that always

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Question

Does even isometric equality hold?

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- *In general, $\left(\mathcal{A}^{min\,\triangle}, \|\cdot\|_{\mathcal{A}^{min\,\triangle}}\right) \neq \left(\mathcal{A}^{*}, \|\cdot\|_{\mathcal{A}^{*}}\right)$ (due to Banach spaces without AP).*

Conjugate operator ideals IV

A deeper investigation of relations between the Banach ideals $(\mathcal{A}^{\triangle}, \|\cdot\|_{\mathcal{A}^{\triangle}})$ and $(\mathcal{A}^{*}, \|\cdot\|_{\mathcal{A}^{*}})$ needs the analysis of a crucial - and non-trivial - local property, known as “accessibility”.

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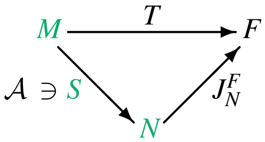
Problem (Grothendieck - 1956)

D'autre part, si on désigne par $\|u\|_\alpha$ *la norme sur* $E \otimes F$ *duale de la norme* $|u'|_{\alpha'}$ *sur* $E' \otimes F'$*, on aura* $\|u\|_\alpha \leq |u|_\alpha$*, mais on ne sait pas si on aura toujours* $\|u\|_\alpha = |u|_\alpha$*.*

Accessible operator ideals I

Definition (Reisner 1979, Defant 1986)

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called right-accessible, if for all $(M, F) \in$ FIN $\times$ BAN, operators $T \in \mathcal{L}(M, F)$ and $\varepsilon > 0$ there are $N \in$ FIN(F) and $S \in \mathcal{L}(M, N)$ such that the following diagram commutes

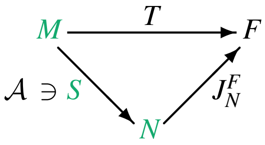


and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.

Quasi-accessible operator ideals I

Definition

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called quasi-right-accessible, if there exists a constant $\kappa \geq 0$ such that for all $(M, F) \in$ FIN $\times$ BAN, operators $T \in \mathcal{L}(M, F)$ and $\varepsilon > 0$ there are $N \in$ FIN(F) and $S \in \mathcal{L}(M, N)$ such that the following diagram commutes

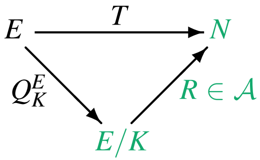


and $\|S\|_{\mathcal{A}} \leq (1 + \varepsilon)\kappa\|T\|_{\mathcal{A}}$.

Accessible operator ideals II

Definition (Reisner 1979, Defant 1986)

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called left-accessible, if for all $(E, N) \in$ BAN $\times$ FIN, operators $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $K \in$ COFIN(E) and $R \in \mathcal{L}(E/K, N)$ such that the following diagram commutes



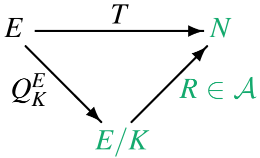
and $\|R\|_{\mathcal{A}} \leq (1 + \varepsilon)\|T\|_{\mathcal{A}}$.

A left- and right-accessible p-Banach ideal is called *accessible*.

Quasi-accessible operator ideals II

Definition

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called quasi-left-accessible, if there exists a constant $\kappa \geq 0$ such that for all $(E, N) \in$ BAN $\times$ FIN, operators $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $K \in$ COFIN(E) and $R \in \mathcal{L}(E/K, N)$ such that the following diagram commutes



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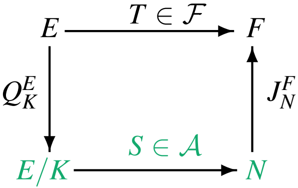
A quasi-left- and quasi-right-accessible p-Banach ideal is called *quasi-accessible*.

Accessible operator ideals III

Definition (Reisner 1979, Defant 1986)

$(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is totally accessible, if for every finite rank operator $T \in \mathcal{F}(E,F)$ between Banach spaces and $\varepsilon > 0$ there are $(K,N) \in \text{COFIN}(E) \times \text{FIN}(F)$ and $S \in \mathcal{L}(E/K, N)$ such that the following diagram commutes



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Quasi-accessible operator ideals III

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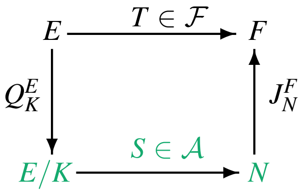

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Obviously, every (quasi-)totally accessible p-Banach ideal is (quasi-)right- and (quasi-)left-accessible ($0 < p \leq 1$).

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Accessible maximal Banach ideals II

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Theorem (Pisier - 1993)

There exists a maximal Banach ideal $(\mathcal{A}_P, \|\cdot\|_{\mathcal{A}_P})$ *which neither is right-accessible nor left-accessible. Moreover,* $(\mathcal{A}_P^{inj}, \|\cdot\|_{\mathcal{A}_P^{inj}})$ *is not left-accessible.*

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There exists a maximal Banach ideal $(\mathcal{A}_P, \|\cdot\|_{\mathcal{A}_P})$ *which neither is right-accessible nor left-accessible. Moreover,* $(\mathcal{A}_P^{inj}, \|\cdot\|_{\mathcal{A}_P^{inj}})$ *is not left-accessible.*

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

- $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *is right-accessible;*
- $(\mathcal{A}^* \circ \mathcal{A}, \|\cdot\|_{\mathcal{A}^* \circ \mathcal{A}}) \subseteq (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$*;*
- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{L}(M, F)$ *and for all* $(M, F) \in$ *FIN* $\times$ *BAN;*

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- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{F}(E_0, F)$*, and for all Banach space pairs* (E_0, F)*, such that* E_0' *has the MAP;*

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal. TFAE:

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- $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is left-accessible.*

Accessible maximal Banach ideals III

Corollary

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a right-accessible maximal Banach ideal. Let* $E_0 \in$ *BAN such that* $Id_{E_0} \in \mathcal{A}(E_0, E_0)$. *Then*

$$\mathcal{A}^*(E_0, F) \subseteq \mathcal{I}(E_0, F)$$

for all Banach spaces F, *and* $\|T\|_{\mathcal{I}} \leq \|Id_{E_0}\|_{\mathcal{A}} \|T\|_{\mathcal{A}^*}$ *for all* $T \in \mathcal{A}^*(E_0, F)$.

Accessible maximal Banach ideals III

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Corollary

Let $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ *be a right-accessible maximal Banach ideal such that* $\left(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*}\right) \subseteq \left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$. *Let* $E_0 \in$ *BAN such that* $Id_{E_0} \in \mathcal{A}^*(E_0, E_0)$. *Then* E_0 *has the BAP (with constant* $\|Id_{E_0}\|_{\mathcal{A}^*}$*).*

Accessible maximal Banach ideals IV

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

- $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is totally-accessible;*
- $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}})$.

Accessible maximal Banach ideals IV

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Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

- $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is totally-accessible;*
- $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}})$.

Corollary (Defant/Floret - 1993)

Let E *be a Banach space such that* $Id_{E_0} \in \mathcal{A}(E_0, E_0)$. *If the adjoint* $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is totally-accessible,* E_0 *must have the BAP (with constant* $\|Id_{E_0}\|_{\mathcal{A}}$*).*

Accessible maximal Banach ideals V

Proposition

Fix an arbitrary Banach space E_0*. Let* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *and* $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *be two arbitrary 1-Banach ideals, such that* $\mathcal{A}(E_0, \cdot) \subseteq \mathcal{B}(E_0, \cdot)$ *and* $\|S\|_{\mathcal{B}} \leq \|S\|_{\mathcal{A}}$ *for all* $S \in \mathcal{A}(E_0, \cdot)$*. If* $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *is left-accessible, then* $\mathcal{B}^*(\cdot, E_0) \subseteq \mathcal{A}^*(\cdot, E_0)$*, and* $\|T\|_{\mathcal{A}^*} \leq \|T\|_{\mathcal{B}^*}$ *for all* $T \in \mathcal{B}^*(\cdot, E_0)$*.*

Accessible maximal Banach ideals V

Proposition

Fix an arbitrary Banach space F_0. Let $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ and $\left(\mathcal{B}, \|\cdot\|_{\mathcal{B}}\right)$ be two arbitrary 1-Banach ideals, such that $\mathcal{A}(\cdot, F_0) \subseteq \mathcal{B}(\cdot, F_0)$ and $\|S\|_{\mathcal{B}} \leq \|S\|_{\mathcal{A}}$ for all $S \in \mathcal{A}(\cdot, F_0)$. If $\left(\mathcal{B}, \|\cdot\|_{\mathcal{B}}\right)$ is right-accessible, then $\mathcal{B}^*(F_0, \cdot) \subseteq \mathcal{A}^*(F_0, \cdot)$, and $\|T\|_{\mathcal{A}^*} \leq \|T\|_{\mathcal{B}^*}$ for all $T \in \mathcal{B}^*(F_0, \cdot)$.

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Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal.

- *If $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$, then $(\mathcal{A}^{*\,inj}, \|\cdot\|_{\mathcal{A}^{*\,inj}})$ is quasi-totally accessible;*

Accessible maximal Banach ideals V

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal.

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- *If $\mathcal{P}_1^{dual} \subseteq \mathcal{A} \subseteq \mathcal{L}_2$, then $(\mathcal{A}^{inj}, \|\cdot\|_{\mathcal{A}^{inj}})$ is quasi-totally accessible.*

Accessible maximal Banach ideals VI

Let $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ be a maximal Banach ideal. Looking at the "increasing sequence"

$$\left(\mathcal{A}^{min}, \|\cdot\|_{\mathcal{A}^{min}}\right) \subseteq \left(\overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}, \|\cdot\|_{\mathcal{A}^{*\triangle}}\right) \subseteq \left(\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}}\right) \subsetneq \left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right),$$

Pisier's counterexample and the accessibility of the "small" ideal $\left(\mathcal{A}^{min}, \|\cdot\|_{\mathcal{A}^{min}}\right)$ lead to a very natural, yet non-trivial question: is $\left(\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}}\right)$ accessible?

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Pisier's counterexample and the accessibility of the "small" ideal $(\mathcal{A}^{min}, \|\cdot\|_{\mathcal{A}^{min}})$ lead to a very natural, yet non-trivial question: is $(\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}})$ accessible? The solution of this question will lead us to a completely different topic; namely:

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A common denominator?

Let us have a quick glimpse at the following non-trivial results:

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- $\left(\overline{\mathcal{F}}, \|\cdot\|\right) = \left(\overline{\mathcal{F}}^{dual}, \|\cdot\|\right) = \left(\overline{\mathcal{F}}^{reg}, \|\cdot\|\right)$
- *Let E be a real Banach space. If the non-empty intersection of two closed balls has a center of symmetry, and if $\dim(E) = k < \infty$, then E is isometric to l_∞^k.*

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Do the proofs of these statements reveal a common denominator?

The strong principle of local reflexivity I

In fact, all proofs are based on the strong principle of local reflexivity (LRP), coined by Johnson, Lindenstrauss, Rosenthal and Zippin [1969-1971] (which - very roughly speaking - states that every Banach space F is "finitely representable" in its bidual F''):

Theorem (Strong principle of local reflexivity (S-LRP))

Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in$ FIN(F'), $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. Then there exists an operator $S \in \mathcal{L}(M, F)$ such that

(i) $\|S\| \leq (1+\varepsilon)\|T\|$;

(ii) $\langle Sx, b\rangle = \langle b, Tx\rangle$ *for all* $(x, b) \in M \times N$;

(iii) $j_F Sx = Tx$ *for all* $x \in M$, *satisfying* $Tx \in j_F(F)$.

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As we will recognise, statements (i) and (ii) are equivalent to the following statement:

The strong principle of local reflexivity II

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Proposition

$(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) = (\mathcal{L}^{\triangle}, \|\cdot\|_{\mathcal{L}^{\triangle}})$ *is left-accessible.*

S-LRP for maximal Banach ideals I

Suppose that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an arbitrary maximal Banach ideal. Is then the following transfer of the strong LRP to (the norm of) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ satisfied?

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Natural Question

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in$ FIN(F'), $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. Does there exist an operator $S \in \mathcal{L}(M, F)$ such that

(i) $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$,

(ii) $\langle Sx, b\rangle = \langle b, Tx\rangle$ *for all* $(x, b) \in M \times N$,

(iii) $j_F Sx = Tx$ *for all* $x \in M$*, satisfying* $Tx \in j_F(F)$*?*

S-LRP for maximal Banach ideals II

Definition

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in \text{FIN}(F')$, $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. If there exist an operator $S \in \mathcal{L}(M, F)$ such that

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we say that the maximal Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ satisfies the $\|\cdot\|_{\mathcal{A}}$-weak principle of local reflexivity.

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We know that the $\|\cdot\|_{\mathcal{A}}$-weak principle of local reflexivity already implies that $j_F Sx = Tx$ for all $x \in M$, satisfying $Tx \in j_F(F)$ (i. e., point (iii) above).

S-LRP for maximal Banach ideals III

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S-LRP for maximal Banach ideals III

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S-LRP for maximal Banach ideals III

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. TFAE:*

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. Then* $(\mathcal{A}^{\triangle dd}, \|\cdot\|_{\mathcal{A}^{\triangle dd}})$ *is accessible.*

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In fact, we have:

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. Then* $(\mathcal{A}^{\triangle}, \|\cdot\|_{\mathcal{A}^{\triangle}})$ *is accessible!*

S-LRP for maximal Banach ideals VII

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Theorem (S-LRP for maximal Banach ideals)

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. Let* F *be an arbitrary Banach space,* $M \in$ *FIN an arbitrary finite dimensional space,* $N \in$ *FIN*(F'), $T \in \mathcal{L}(M, F'')$, *and* $\varepsilon > 0$. *Then there exists an operator* $S \in \mathcal{L}(M, F)$ *such that*

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An approximation result

Let us conclude this presentation with the following non-trivial application of the LRP for maximal Banach ideals:

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. Let* E *and* F *be Banach spaces. Suppose that one of the following conditions is satisfied:*

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(i) *E' has MAP;*

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Then

$$\mathcal{A}^{min}(E,F) = \overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}(E,F) \overset{1}{\hookrightarrow} \mathcal{A}^{*\triangle}(E,F),$$

and $\|T\|_{\mathcal{A}^{*\triangle}} = \|T\|_{\mathcal{A}^{min}}$ *for every* $T \in \overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}(E,F)$.

- Suppose $T \in \mathcal{N}^{\triangle}(E, F)$, where (E, F) are arbitrary Banach spaces. Does T factor through a Banach space which has the AP?

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- Is the following isometric equality true:

$$(\mathcal{K} \circ \mathcal{N}^{\triangle}, \|\cdot\|_{\mathcal{K} \circ \mathcal{N}^{\triangle}}) \stackrel{(?)}{=} (\overline{\mathcal{F}}, \|\cdot\|) \stackrel{(?)}{=} (\mathcal{N}^{\triangle} \circ \mathcal{K}, \|\cdot\|_{\mathcal{N}^{\triangle} \circ \mathcal{K}})\,?$$

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- Describe useful relations between $\left(\mathcal{N}^{\triangle \text{ dual}}, \|\cdot\|_{\mathcal{N}^{\triangle \text{ dual}}}\right)$ and $\left(\mathcal{I}^{\triangle \text{ dual}}, \|\cdot\|_{\mathcal{I}^{\triangle \text{ dual}}}\right)$!

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Thank you for your attention!

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Are there any questions, comments or remarks?