Approximation properties of Banach spaces, the principle of local reflexivity for operator ideals, and factorisation of operators with finite rank

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Here, we would like to add the following point: it has been demonstrated that the already very powerful analytic theory of operator ideals can be significantly strengthened by implementing the no less powerful theory of tensor products of Banach spaces (including its inherent algebraic structure). It all starts with A. Grothendieck's canonical isometric isomorphism:

$$\mathcal{L}^2(E,F) \cong \mathcal{L}(E,F') \overset{(!)}{\cong} G'$$

where $E, F \in BAN$ and $G := E\widetilde{\otimes}_\pi F$.

Operator ideals: definition I

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An operator ideal $\mathcal{A}$ is an assignment of ascribing to each pair of Banach spaces (E,F) a linear subspace $\mathcal{A}(E,F)$ of $\mathcal{L}(E,F)$ satisfying the following conditions

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(I1) $Id_{\mathbb{K}} \in \mathcal{A}(\mathbb{K},\mathbb{K})$;

(I2) If E_0 and F_0 are Banach spaces, then $RST \in \mathcal{A}(E_0,F_0)$, whenever $T \in \mathcal{L}(E_0,E), S \in \mathcal{A}(E,F)$ and $R \in \mathcal{L}(F,F_0)$.

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Definition (ctd.)

Further, if for each pair of Banach spaces (E, F), $\mathcal{A}(E, F)$ is supplied with a norm $\|\cdot\|_{\mathcal{A}}$, satisfying

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(N3) $\left(\mathcal{A}(E,F), \|\cdot\|_{\mathcal{A}}\right)$ is a Banach space,

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Operator ideals: definition III

Definition (ctd.)

Further, if for each pair of Banach spaces (E, F), $\mathcal{A}(E, F)$ is supplied with a p-norm $\|\cdot\|_{\mathcal{A}}$, with $0 < p \leq 1$, satisfying

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then the pair $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called a p-normed operator ideal. If in addition

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Banach ideals: important examples I

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- $(\mathcal{W}, \|\cdot\|)$: the Banach ideal consisting of all weakly compact operators (coinciding with the class of all those bounded linear operators between Banach spaces which factor through a reflexive Banach space).

Banach ideals: important examples II

- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$: the smallest *Banach* ideal consisting of the class of all nuclear operators between Banach spaces:

  $T \in \mathcal{N}(E,F)$ iff there exist sequences $(a_n)_{n\in\mathbb{N}} \subseteq E'$ and $(y_n)_{n\in\mathbb{N}} \subseteq F$ such that $\sum_{n=1}^{\infty} \|a_n\| \|y_n\| < \infty$ and $T = \sum_{n=1}^{\infty} \langle \cdot, a_n \rangle y_n = \sum_{n=1}^{\infty} a_n \underline{\otimes} y_n$, implying that $\|T\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|a_n\| \|y_n\| : T = \dots \right\}$ is well-defined.

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- $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$: the Banach ideal of all integral operators:

  $T \in \mathcal{I}(E, F)$ iff there exists a constant $c \geq 0$ such that for all finite rank operators $L \in \mathcal{F}(F, E)$

  $$|\mathrm{tr}(TL)| \leq c \|L\| .$$

  $\|T\|_{\mathcal{I}} := \inf\{c : c \text{ satisfies } \dots\}$ is well-defined for any integral operator T.

Banach ideals: important examples III

Theorem (Grothendieck – 1956)

$T \in \mathcal{I}(E,F)$ if and only if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and operators $S \in \mathcal{L}(E, L^\infty(\mathbb{P}))$, $R \in \mathcal{L}\big(L^1(\mathbb{P}), F''\big)$, such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "j_F T"] \arrow[d, "S"'] & F'' \\ L^\infty(\mathbb{P}) \arrow[r, "J^{L^1(\mathbb{P})}_{L^\infty(\mathbb{P})}"] & L^1(\mathbb{P}) \arrow[u, "R"'] \end{tikzcd}$$

and $\|T\|_{\mathcal{I}} = \inf\{\|R\|\,\|S\| \;:\; \dots\}$, where the infimum is taken over all possible $\mathbb{P}$'s, R's, and S's.

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and $\|T\|_{\mathcal{I}} = \inf\{\|R\|\,\|S\| \ : \ldots\}$, where the infimum is taken over all possible $\mathbb{P}$'s, R's, and S's.

Main Idea of a Proof (Grothendieck + tensor norms !)

Put $\Omega := B_{E'} \times B_{F''}$. Then $E \otimes_\varepsilon F' \overset{1}{\hookrightarrow} L^\infty(\Omega, \mathbb{P}) \otimes_\varepsilon (L^1(\Omega, \mathbb{P}))'$...

Banach ideals: important examples IV

- Fix $1 \leq p < \infty$, and consider $(\mathcal{P}_p, \|\cdot\|_{\mathcal{P}_p})$, the Banach ideal of all absolutely p-summing operators; i. e., the class of all operators between Banach spaces which map weakly p-summable sequences in E to strongly p-summable sequences in F.

Banach ideals: important examples IV

- Fix $1 \leq p < \infty$, and consider $\left(\mathcal{P}_p, \|\cdot\|_{\mathcal{P}_p}\right)$, the Banach ideal of all absolutely p-summing operators; i. e., the class of all operators between Banach spaces which map weakly p-summable sequences in E to strongly p-summable sequences in F. Hence, $T \in \mathcal{P}_p(E, F)$ iff there exists a constant $c \geq 0$ such that for all $n \in \mathbb{N}$, and for all $(x_1, \ldots, x_n) \in E^n$

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{\frac{1}{p}} \leq c \sup\left\{\Big(\sum_{k=1}^{n} |\langle x_k, a\rangle|^p\Big)^{\frac{1}{p}} : a \in B_{E'}\right\},$$

implying that $\|T\|_{\mathcal{P}_p} := \inf\{c : c \text{ satisfies } \ldots\}$ is well-defined for any absolutely p-summing operator T.

Banach ideals: important examples V

- $\left(\mathcal{L}_2, \|\cdot\|_{\mathcal{L}_2}\right)$: the Banach ideal consisting of all operators between Banach spaces which factor through a Hilbert space:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\mathcal{L} \ni S"'] & & F \\ & H \arrow[ur, "R \in \mathcal{L}"'] & \end{tikzcd}$$

$\|T\|_{\mathcal{L}_2} := \inf\{\|R\| \, \|S\| \ : T = RS \ldots\}$, where the infimum is taken over all possible Hilbert spaces H and factorising R's and S's.

Banach ideals: important examples VI

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- $(\mathcal{L}_\infty, \|\cdot\|_{\mathcal{L}_\infty})$: the Banach ideal consisting of all operators between Banach spaces which factor through some $C(K)$, where K is a compact set, in the following sense:

$$
\begin{tikzcd}
E \arrow[rr, "j_F T"] \arrow[dr, "\mathcal{L} \ni S"'] & & F'' \\
& C(K) \arrow[ur, "R \in \mathcal{L}"'] &
\end{tikzcd}
$$

$\|T\|_{\mathcal{L}_\infty} := \inf\{\|R\| \, \|S\| \ : T = RS \ldots\}$, where the infimum is taken over all possible spaces $C(K)$ (K compact) and factorising R's and S's.

Banach ideals: important examples VII

- $(\mathcal{L}_1, \|\cdot\|_{\mathcal{L}_1})$: the Banach ideal consisting of all operators between Banach spaces which factor through some $L^1(\mu)$, where μ is a Borel-Radon measure, in the following sense:

$$\begin{tikzcd}
E \arrow[rr, "j_F T"] \arrow[dr, "\mathcal{L} \ni S"'] & & F'' \\
& L^1(\mu) \arrow[ur, "R \in \mathcal{L}"'] &
\end{tikzcd}$$

$\|T\|_{\mathcal{L}_1} := \inf\{\|R\|\,\|S\| \ : T = RS \ldots\}$, where the infimum is taken over all possible spaces $L^1(\mu)$ (μ a Borel-Radon measure) and factorising R's and S's.

In the following, if not differently stated, let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ always be an arbitrary p-Banach ideal and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ always an arbitrary q-Banach ideal, where $0 < p, q \leq 1$.

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If an arbitrary operator ideal is given, one can "derive canonically" further operator ideals, such as e. g. the following very important candidates:

Dual operator ideal and regular hull

- $\left(\mathcal{A}^{\text{dual}}, \|\cdot\|_{\mathcal{A}^{\text{dual}}}\right)$: the dual p-Banach ideal. $T \in \mathcal{A}^{\text{dual}}(E, F)$ iff $T' \in \mathcal{A}(F', E')$; $\|T\|_{\mathcal{A}^{\text{dual}}} := \|T'\|_{\mathcal{A}}$;

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- $(\mathcal{A}^{\text{reg}}, \|\cdot\|_{\mathcal{A}^{\text{reg}}})$: the regular hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. $T \in \mathcal{A}^{\text{reg}}(E,F)$ iff $j_F T \in \mathcal{A}(E,F'')$, where $j_F : F \overset{1}{\hookrightarrow} F''$ is the canonical injection. $\|T\|_{\mathcal{A}^{\text{reg}}} := \|j_F T\|_{\mathcal{A}}$;

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Recall that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is regular iff $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{reg}}, \|\cdot\|_{\mathcal{A}^{\text{reg}}})$.

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- $(\mathcal{K}^{dual}, \|\cdot\|) = (\mathcal{K}, \|\cdot\|)$;
- $(\mathcal{P}_1, \|\cdot\|_{\mathcal{P}_1}) = (\mathcal{P}_1^{reg}, \|\cdot\|_{\mathcal{P}_1^{reg}}) \subsetneq (\mathcal{L}_2, \|\cdot\|_{\mathcal{L}_2}) = (\mathcal{L}_2^{reg}, \|\cdot\|_{\mathcal{L}_2^{reg}}) = (\mathcal{L}_2^{dual}, \|\cdot\|_{\mathcal{L}_2^{dual}})$.

Injective and surjective operator ideals

- $\left(\mathcal{A}^{\text{inj}}, \|\cdot\|_{\mathcal{A}^{\text{inj}}}\right)$: the injective hull of $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$. $T \in \mathcal{A}^{\text{inj}}(E, F)$ iff $J_F T \in \mathcal{A}(E, F^\infty)$, where $F^\infty := C(B_{F'})$ and $J_F : F \overset{1}{\hookrightarrow} F^\infty$ is the canonical isometric embedding. $\|T\|_{\mathcal{A}^{\text{inj}}} := \|J_F T\|_{\mathcal{A}}$;

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- $(\mathcal{A}^{\text{sur}}, \|\cdot\|_{\mathcal{A}^{\text{sur}}})$: the surjective hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$.
  $T \in \mathcal{A}^{\text{sur}}(E, F)$ iff $TQ_E \in \mathcal{A}(E^1, F)$, where $E^1 := l_1(B_E, \mathbb{K})$ and $Q_E : E^1 \overset{1}{\twoheadrightarrow} E$ is the canonical metric surjection.
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Recall that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is injective (respectively surjective) iff $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{inj}}, \|\cdot\|_{\mathcal{A}^{\text{inj}}})$ (resp. $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{sur}}, \|\cdot\|_{\mathcal{A}^{\text{sur}}})$).

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Examples

Injective and surjective operator ideals

- $(\mathcal{A}^{\text{inj}}, \|\cdot\|_{\mathcal{A}^{\text{inj}}})$: the injective hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. $T \in \mathcal{A}^{\text{inj}}(E,F)$ iff $J_F T \in \mathcal{A}(E, F^\infty)$, where $F^\infty := C(B_{F'})$ and $J_F : F \overset{1}{\hookrightarrow} F^\infty$ is the canonical isometric embedding. $\|T\|_{\mathcal{A}^{\text{inj}}} := \|J_F T\|_{\mathcal{A}}$;
- $(\mathcal{A}^{\text{sur}}, \|\cdot\|_{\mathcal{A}^{\text{sur}}})$: the surjective hull of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. $T \in \mathcal{A}^{\text{sur}}(E,F)$ iff $TQ_E \in \mathcal{A}(E^1, F)$, where $E^1 := l_1(B_E, \mathbb{K})$ and $Q_E : E^1 \overset{1}{\twoheadrightarrow} E$ is the canonical metric surjection. $\|T\|_{\mathcal{A}^{\text{sur}}} := \|TQ_E\|_{\mathcal{A}}$;

Recall that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is injective (respectively surjective) iff $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{inj}}, \|\cdot\|_{\mathcal{A}^{\text{inj}}})$ (resp. $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{\text{sur}}, \|\cdot\|_{\mathcal{A}^{\text{sur}}})$).

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Products of operator ideals I

$T \in \mathcal{L}(E,F)$ is an element of the product ideal $\mathcal{A} \circ \mathcal{B}(E,F)$ iff there exists a Banach space G and operators $R \in \mathcal{B}(E,G)$ and $S \in \mathcal{A}(G,F)$ such that $T = SR$:

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$(\mathcal{A} \circ \mathcal{B}, \|\cdot\|_{\mathcal{A} \circ \mathcal{B}})$ *is a* r*-Banach ideal, where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*. In general,* $\|\cdot\|_{\mathcal{A} \circ \mathcal{B}}$ *is not a norm.*

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Products of operator ideals III

Theorem (Grothendieck's inequality in operator form)

Every (bounded linear) operator from l_1 to l_2 is absolutely 1-summing, and

$$\|T\|_{\mathcal{P}_1} \leq K_G \|T\|$$

for some (universal) constant $K_G > 0$.

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Let us recall the following - now naturally appearing - definition:

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A Banach space G is called Grothendieck space - or short "GT space" if every (bounded linear) operator from G to l_2 is absolutely 1-summing;

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There exists a constant $c_G \geq 0$ such that $\|T\|_{\mathcal{P}_1} \leq c_G \|T\|$. In particular, we have $Id_G \in (\mathcal{L}_\infty \circ \mathcal{L}_2)^*(G, G)$, and $\|Id_G\|_{(\mathcal{L}_\infty \circ \mathcal{L}_2)^*} \leq c_G$.

Adjoint operator ideal I

Let us recall that $T \in \mathcal{A}^*(E, F)$ iff there exists a constant $c \geq 0$ such that

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for all Banach spaces $M \in \mathrm{FIN}(E)$, $K \in \mathrm{COFIN}(F)$, and $S \in \mathcal{L}(F/K, M)$. If we denote

$$\|T\|_{\mathcal{A}^*} := \inf(c),$$

where the infimum is taken over all such constants c, we obtain a Banach ideal $\left(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*}\right)$, the adjoint of $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$.

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Theorem

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Corollary

Let $\mathcal{B}$ be a maximal Banach ideal. If $\mathcal{B} \circ \mathcal{L}_2$ is a Banach ideal, then $\mathcal{B}^*$ cannot be injective.

Minimal kernel and compact kernel

Of particular importance are the following two product ideal constructions:

- The minimal kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$:

$$(\mathcal{A}^{\mathsf{min}}, \|\cdot\|_{\mathcal{A}^{\mathsf{min}}}) := (\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}, \| \cdot \|_{\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}})$$

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- $(\overline{\mathcal{F}}, \|\cdot\|) = (\mathcal{L}^{min}, \|\cdot\|_{\mathcal{L}^{min}}) = (\overline{\mathcal{F}}^{com}, \|\cdot\|_{\overline{\mathcal{F}}^{com}})$;

Minimal kernel and compact kernel

Of particular importance are the following two product ideal constructions:

- The minimal kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$:

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- The compact kernel of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ [Karn-Sinha – 2012]:

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Examples

- $\left(\overline{\mathcal{F}}, \|\cdot\|\right) = \left(\mathcal{L}^{min}, \|\cdot\|_{\mathcal{L}^{min}}\right) = \left(\overline{\mathcal{F}}^{com}, \|\cdot\|_{\overline{\mathcal{F}}^{com}}\right)$;
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Examples

- $(\overline{\mathcal{F}}, \|\cdot\|) = (\mathcal{L}^{min}, \|\cdot\|_{\mathcal{L}^{min}}) = (\overline{\mathcal{F}}^{com}, \|\cdot\|_{\overline{\mathcal{F}}^{com}})$;
- $(\mathcal{N}, \|\cdot\|_{\mathcal{N}}) = (\mathcal{I}^{min}, \|\cdot\|_{\mathcal{I}^{min}})$;
- *Let* $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ *be a Banach ideal. Then* $(\mathcal{C}^* \circ \mathcal{C}^{min}, \| \cdot \|_{\mathcal{C}^* \circ \mathcal{C}^{min}}) \subseteq (\mathcal{N}, \|\cdot\|_{\mathcal{N}})$.

Conjugate operator ideals I

$T \in \mathcal{A}^{\triangle}(E,F)$ iff there exists a constant $c \geq 0$ such that for all finite rank operators $L \in \mathcal{F}(F,E)$

$$|\mathrm{tr}(TL)| \leq c \cdot \|L\|_{\mathcal{A}}$$

If we put

$$\|T\|_{\mathcal{A}^{\triangle}} := \inf(c),$$

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- $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) = (\mathcal{N}^{\triangle\triangle}, \|\cdot\|_{\mathcal{N}^{\triangle\triangle}}) = (\overline{\mathcal{F}}^{\triangle}, \|\cdot\|_{\overline{\mathcal{F}}^{\triangle}})$.

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It is trivial to see that always

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Does even isometric equality hold?

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Question

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Observation

Let E and F be arbitrary Banach spaces. Let $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ and $\left(\mathcal{B}, \|\cdot\|_{\mathcal{B}}\right)$ be as before. If $\mathcal{A}(E,F) \subseteq \mathcal{B}(E,F)$ and $\|S\|_{\mathcal{B}} \leq \|S\|_{\mathcal{A}}$ for all $S \in \mathcal{F}(E,F)$, then $\mathcal{B}^{\triangle}(F,E) \subseteq \mathcal{A}^{\triangle}(F,E)$, and $\|T\|_{\mathcal{A}^{\triangle}} \leq \|T\|_{\mathcal{B}^{\triangle}}$ for all $T \in \mathcal{B}^{\triangle}(F,E)$.

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- $\left(\mathcal{L}_\infty^{\triangle}, \|\cdot\|_{\mathcal{L}_\infty^{\triangle}}\right) \overset{(!)}{\neq} \left(\mathcal{L}_\infty^{*}, \|\cdot\|_{\mathcal{L}_\infty^{*}}\right) = \left(\mathcal{P}_1, \|\cdot\|_{\mathcal{P}_1}\right)$;
- *In general, $\left(\mathcal{A}^{min\,\triangle}, \|\cdot\|_{\mathcal{A}^{min\,\triangle}}\right) \neq \left(\mathcal{A}^{*}, \|\cdot\|_{\mathcal{A}^{*}}\right)$ (due to Banach spaces without AP). However, the following inclusion is always satisfied:*

$$\left(\mathcal{N}^{\triangle} \circ \mathcal{A}^{*}, \|\cdot\|_{\mathcal{N}^{\triangle} \circ \mathcal{A}^{*}}\right) \subseteq \left(\mathcal{A}^{min\,\triangle}, \|\cdot\|_{\mathcal{A}^{min\,\triangle}}\right).$$

Conjugate operator ideals IV

A deeper investigation of relations between the Banach ideals $(\mathcal{A}^{\triangle}, \|\cdot\|_{\mathcal{A}^{\triangle}})$ and $(\mathcal{A}^{*}, \|\cdot\|_{\mathcal{A}^{*}})$ needs the analysis of a crucial - and non-trivial - local property, known as “accessibility”.

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Related research started with the following problem formulation of Grothendieck in his famous *RÉSUMÉ DE LA THÉORIE MÉTRIQUE DES PRODUITS TENSORIELS TOPOLOGIQUES*:

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Problem (Grothendieck - 1956)

D’autre part, si on désigne par $\|u\|_{\alpha}$ la norme sur $E \otimes F$ duale de la norme $|u'|_{\alpha'}$ sur $E' \otimes F'$, on aura $\|u\|_{\alpha} \leq |u|_{\alpha}$, mais on ne sait pas si on aura toujours $\|u\|_{\alpha} = |u|_{\alpha}$.

Accessible operator ideals I

Definition (Reisner 1979, Defant 1986)

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called right-accessible, if for all $(M, F) \in$ FIN $\times$ BAN, operators $T \in \mathcal{L}(M, F)$ and $\varepsilon > 0$ there are $N \in$ FIN(F) and $S \in \mathcal{L}(M, N)$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[rr, "T"] \arrow[dr, "\mathcal{A} \ni S"'] & & F \\ & N \arrow[ur, "J_N^F"'] & \end{tikzcd}$$

and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.

Quasi-accessible operator ideals I

Definition

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called quasi-right-accessible, if there exists a constant $\kappa \geq 0$ such that for all $(M, F) \in$ FIN $\times$ BAN, operators $T \in \mathcal{L}(M, F)$ and $\varepsilon > 0$ there are $N \in$ FIN(F) and $S \in \mathcal{L}(M, N)$ such that the following diagram commutes

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Accessible operator ideals II

Definition (Reisner 1979, Defant 1986)

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called left-accessible, if for all $(E, N) \in$ BAN $\times$ FIN, operators $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $K \in$ COFIN(E) and $R \in \mathcal{L}(E/K, N)$ such that the following diagram commutes

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A left- and right-accessible p-Banach ideal is called *accessible*.

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Definition

Let $0 < p \leq 1$. A p-Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called quasi-left-accessible, if there exists a constant $\kappa \geq 0$ such that for all $(E, N) \in$ BAN $\times$ FIN, operators $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $K \in$ COFIN(E) and $R \in \mathcal{L}(E/K, N)$ such that the following diagram commutes

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A quasi-left- and quasi-right-accessible p-Banach ideal is called *quasi-accessible*.

Accessible operator ideals III

Definition (Reisner 1979, Defant 1986)

$(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is totally accessible, if for every finite rank operator $T \in \mathcal{F}(E,F)$ between Banach spaces and $\varepsilon > 0$ there are $(K,N) \in \mathrm{COFIN}(E) \times \mathrm{FIN}(F)$ and $S \in \mathcal{L}(E/K, N)$ such that the following diagram commutes

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Definition

$(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is quasi-totally accessible, if there exists a constant $\kappa \geq 0$ such that for every finite rank operator $T \in \mathcal{F}(E, F)$ between Banach spaces and $\varepsilon > 0$ there are $(K, N) \in \mathrm{COFIN}(E) \times \mathrm{FIN}(F)$ and $S \in \mathcal{L}(E/K, N)$ such that the following diagram commutes

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Theorem (Pisier - 1993)

There exists a maximal Banach ideal $(\mathcal{A}_P, \|\cdot\|_{\mathcal{A}_P})$ *which neither is right-accessible nor left-accessible. Moreover,* $(\mathcal{A}_P^{inj}, \|\cdot\|_{\mathcal{A}_P^{inj}})$ *is not left-accessible.*

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- $(\mathcal{A}^* \circ \mathcal{A}, \|\cdot\|_{\mathcal{A}^* \circ \mathcal{A}}) \subseteq (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$;
- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{L}(M, F)$ *and for all* $(M, F) \in$ *FIN* $\times$ *BAN;*
- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{F}(E_0, F)$, *and for all Banach space pairs* (E_0, F), *such that* E_0' *has the MAP;*

Accessible maximal Banach ideals II

Theorem (Pisier - 1993)

There exists a maximal Banach ideal $(\mathcal{A}_P, \|\cdot\|_{\mathcal{A}_P})$ *which neither is right-accessible nor left-accessible. Moreover,* $(\mathcal{A}_P^{inj}, \|\cdot\|_{\mathcal{A}_P^{inj}})$ *is not left-accessible.*

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

- $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *is right-accessible;*
- $(\mathcal{A}^* \circ \mathcal{A}, \|\cdot\|_{\mathcal{A}^* \circ \mathcal{A}}) \subseteq (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$;
- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{L}(M, F)$ *and for all* $(M, F) \in$ *FIN* $\times$ *BAN;*
- $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{*\triangle}}$ *for all* $T \in \mathcal{F}(E_0, F)$*, and for all Banach space pairs* (E_0, F)*, such that* E_0' *has the MAP;*
- $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is left-accessible.*

Accessible maximal Banach ideals III

Corollary

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a right-accessible maximal Banach ideal. Let $E_0 \in$ BAN such that $Id_{E_0} \in \mathcal{A}(E_0, E_0)$. Then

$$\mathcal{A}^*(E_0, F) \subseteq \mathcal{I}(E_0, F)$$

for all Banach spaces F, and $\|T\|_{\mathcal{I}} \leq \|Id_{E_0}\|_{\mathcal{A}} \|T\|_{\mathcal{A}^*}$ for all $T \in \mathcal{A}^*(E_0, F)$.

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Corollary

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a right-accessible maximal Banach ideal such that $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*}) \subseteq (\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. Let $E_0 \in$ BAN such that $Id_{E_0} \in \mathcal{A}^*(E_0, E_0)$. Then E_0 has the BAP (with constant $\|Id_{E_0}\|_{\mathcal{A}^*}$).

Accessible maximal Banach ideals IV

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Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

Accessible maximal Banach ideals IV

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Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

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Accessible maximal Banach ideals IV

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

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- $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}})$.

Accessible maximal Banach ideals IV

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be a maximal Banach ideal. TFAE:*

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Corollary (Defant/Floret - 1993)

Let E *be a Banach space such that* $Id_{E_0} \in \mathcal{A}(E_0, E_0)$*. If the adjoint* $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ *is totally-accessible,* E_0 *must have the BAP (with constant* $\|Id_{E_0}\|_{\mathcal{A}}$*).*

Accessible maximal Banach ideals V

Proposition

Fix an arbitrary Banach space E_0. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be two arbitrary 1-Banach ideals, such that $\mathcal{A}(E_0,\cdot) \subseteq \mathcal{B}(E_0,\cdot)$ and $\|S\|_{\mathcal{B}} \leq \|S\|_{\mathcal{A}}$ for all $S \in \mathcal{A}(E_0,\cdot)$. If $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is left-accessible, then $\mathcal{B}^*(\cdot, E_0) \subseteq \mathcal{A}^*(\cdot, E_0)$, and $\|T\|_{\mathcal{A}^*} \leq \|T\|_{\mathcal{B}^*}$ for all $T \in \mathcal{B}^*(\cdot, E_0)$.

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Proposition

Fix an arbitrary Banach space F_0. Let $\left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ and $\left(\mathcal{B}, \|\cdot\|_{\mathcal{B}}\right)$ be two arbitrary 1-Banach ideals, such that $\mathcal{A}(\cdot, F_0) \subseteq \mathcal{B}(\cdot, F_0)$ and $\|S\|_{\mathcal{B}} \leq \|S\|_{\mathcal{A}}$ for all $S \in \mathcal{A}(\cdot, F_0)$. If $\left(\mathcal{B}, \|\cdot\|_{\mathcal{B}}\right)$ is right-accessible, then $\mathcal{B}^*(F_0, \cdot) \subseteq \mathcal{A}^*(F_0, \cdot)$, and $\|T\|_{\mathcal{A}^*} \leq \|T\|_{\mathcal{B}^*}$ for all $T \in \mathcal{B}^*(F_0, \cdot)$.

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Accessible maximal Banach ideals V

Southampton School of Mathematics

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal.

- *If $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$, then $(\mathcal{A}^{*\,inj}, \|\cdot\|_{\mathcal{A}^{*\,inj}})$ is quasi-totally accessible;*

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- *If $\mathcal{P}_1^{dual} \subseteq \mathcal{A} \subseteq \mathcal{L}_2$, then $(\mathcal{A}^{inj}, \|\cdot\|_{\mathcal{A}^{inj}})$ is quasi-totally accessible.*

Accessible maximal Banach ideals VI

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal. Looking at the "increasing sequence"

$$(\mathcal{A}^{min}, \|\cdot\|_{\mathcal{A}^{min}}) \subseteq (\overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}, \|\cdot\|_{\mathcal{A}^{*\triangle}}) \subseteq (\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}}) \subsetneq (\mathcal{A}, \|\cdot\|_{\mathcal{A}}),$$

Pisier's counterexample and the accessibility of the "small" ideal $(\mathcal{A}^{min}, \|\cdot\|_{\mathcal{A}^{min}})$ lead to a very natural, yet non-trivial question: is $(\mathcal{A}^{*\triangle}, \|\cdot\|_{\mathcal{A}^{*\triangle}})$ accessible?

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A common denominator?

Let us have a quick glimpse at the following non-trivial results:

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A common denominator?

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- $\left(\overline{\mathcal{F}}, \|\cdot\|\right) = \left(\overline{\mathcal{F}}^{dual}, \|\cdot\|\right) = \left(\overline{\mathcal{F}}^{reg}, \|\cdot\|\right)$
- *Let E be a real Banach space. If the non-empty intersection of two closed balls has a center of symmetry, and if $\dim(E) = k < \infty$, then E is isometric to l_∞^k.*

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- *Let E be a Banach space. E has the AP iff $Id_{E'}$ belongs to the weak*-closure of $\mathcal{F}\left(E', E'\right)$.*

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Do the proofs of these statements reveal a common denominator?

The strong principle of local reflexivity I

In fact, all proofs are based on the strong principle of local reflexivity (LRP), coined by Johnson, Lindenstrauss, Rosenthal and Zippin [1969-1971] (which - very roughly speaking - states that every Banach space F is “finitely representable” in its bidual F''):

Theorem (Strong principle of local reflexivity (S-LRP))

Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in \mathrm{FIN}(F')$, $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. Then there exists an operator $S \in \mathcal{L}(M, F)$ such that

(i) $\|S\| \leq (1+\varepsilon)\|T\|$;

(ii) $\langle Sx, b\rangle = \langle b, Tx\rangle$ *for all* $(x, b) \in M \times N$;

(iii) $j_F Sx = Tx$ *for all* $x \in M$, *satisfying* $Tx \in j_F(F)$.

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As we will recognise, statements (i) and (ii) are equivalent to the following statement:

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The strong principle of local reflexivity II

Proposition

$(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) = (\mathcal{L}^{\triangle}, \|\cdot\|_{\mathcal{L}^{\triangle}})$ *is left-accessible.*

The strong principle of local reflexivity II

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Moreover, by implementing a result of Pietsch, we conjecture that the S-LRP actually could be viewed as a more general statement involving the geometry of dual Banach spaces with MAP:

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Moreover, by implementing a result of Pietsch, we conjecture that the S-LRP actually could be viewed as a more general statement involving the geometry of dual Banach spaces with MAP:

Conjecture

Let E_0 be a Banach space, such that E_0' has the MAP, and let F be an arbitrary Banach space, $N \in$ FIN(F'), $T \in \mathcal{F}(E_0, F'')$, and $\varepsilon > 0$. Then there exists an operator $R \in \mathcal{F}(E_0, F)$ such that

(i) $\|R\| \leq (1+\varepsilon)\|T\|$;

(ii) $\langle Rz, b\rangle = \langle b, Tz\rangle$ *for all* $(z, b) \in E_0 \times N$;

(iii) $j_F Rz = Tz$ *for all* $z \in E_0$, *satisfying* $Tz \in j_F(F)$.

S-LRP for maximal Banach ideals I

Suppose that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an arbitrary maximal Banach ideal. Is then the following transfer of the strong LRP to (the norm of) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ satisfied?

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Natural Question

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in$ FIN(F'), $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. Does there exist an operator $S \in \mathcal{L}(M, F)$ such that

(i) $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$,

(ii) $\langle Sx, b\rangle = \langle b, Tx\rangle$ *for all* $(x, b) \in M \times N$,

(iii) $j_F Sx = Tx$ *for all* $x \in M$, *satisfying* $Tx \in j_F(F)$?

S-LRP for maximal Banach ideals II

Definition

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in \text{FIN}(F')$, $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. If there exist an operator $S \in \mathcal{L}(M, F)$ such that

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we say that the maximal Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ satisfies the $\|\cdot\|_{\mathcal{A}}$-weak principle of local reflexivity.

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We know that the $\|\cdot\|_{\mathcal{A}}$-weak principle of local reflexivity already implies that $j_F Sx = Tx$ for all $x \in M$, satisfying $Tx \in j_F(F)$ (i. e., point (iii) above).

S-LRP for maximal Banach ideals III

Theorem

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S-LRP for maximal Banach ideals III

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. TFAE:*

- *The* $\|\cdot\|_{\mathcal{A}}$*-weak principle of local reflexivity is satisfied;*
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Let us recall the following - non-trivial - result from 1991:

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Then $(\mathcal{A}^{\triangle dd}, \|\cdot\|_{\mathcal{A}^{\triangle dd}})$ is accessible.

S-LRP for maximal Banach ideals III

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In fact, we have:

Theorem

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Then $(\mathcal{A}^{\triangle}, \|\cdot\|_{\mathcal{A}^{\triangle}})$ is accessible!

S-LRP for maximal Banach ideals VII

Theorem (S-LRP for maximal Banach ideals)

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an arbitrary maximal Banach ideal. Let F be an arbitrary Banach space, $M \in$ FIN an arbitrary finite dimensional space, $N \in$ FIN(F'), $T \in \mathcal{L}(M, F'')$, and $\varepsilon > 0$. Then there exists an operator $S \in \mathcal{L}(M, F)$ such that

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An approximation result I

Let us conclude this presentation with the following non-trivial application of the LRP for maximal Banach ideals:

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$$\mathcal{A}^{min}(E,F) = \overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}(E,F) \overset{1}{\hookrightarrow} \mathcal{A}^{*\triangle}(E,F),$$

and $\|T\|_{\mathcal{A}^{*\triangle}} = \|T\|_{\mathcal{A}^{min}}$ *for every* $T \in \overline{\mathcal{F}}^{\mathcal{A}^{*\triangle}}(E,F)$.

An approximation result II

The latter result immediately implies the following known statements:

Corollary (Defant/Floret - 1993)

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *be an arbitrary maximal Banach ideal. TFAE:*

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(i) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ *is right-accessible;*

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- Describe useful relations between $(\mathcal{N}^{\triangle\,\text{dual}}, \|\cdot\|_{\mathcal{N}^{\triangle\,\text{dual}}})$ and $(\mathcal{I}^{\triangle\,\text{dual}}, \|\cdot\|_{\mathcal{I}^{\triangle\,\text{dual}}})$!

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Are there any questions, comments or remarks?