
# Mathematical Modelling of Infectious Diseases for Public Health

**Frank Oertel**

Philosophy, Logic & Scientific Method
Centre for Philosophy of Natural and Social Sciences (CPNSS)
London School of Economics & Political Science, UK
http://www.frank-oertel-math.de

*Public Health England, Porton Down, Salisbury - Discussion Meeting*

*14.09.2018*


# Contents


1. Revisiting Deterministic Epidemic Models

2. Catching a Glimpse of Stochastic Epidemic Models


# Simple Kermack-McKendrick SIR Model I

The most "simple" deterministic epidemic model is built on the following strongly simplifying assumptions:


# Simple Kermack-McKendrick SIR Model I

The most "simple" deterministic epidemic model is built on the following strongly simplifying assumptions:

(i) At any time point an individual in the population under consideration is either susceptible $(S)$, infected and infectious $(I)$ or recovered and immune $(R)$.


# Simple Kermack-McKendrick SIR Model I

The most “simple” deterministic epidemic model is built on the following strongly simplifying assumptions:

(i) At any time point an individual in the population under consideration is either susceptible $(S)$, infected and infectious $(I)$ or recovered and immune $(R)$.

(ii) The disease is transmitted by contact between an infected and infectious individual and a susceptible individual. Only susceptible individuals can get infected and, after having been infectious for some time, an individual recovers and becomes completely immune for the remainder of the study period.


# Simple Kermack-McKendrick SIR Model I

The most “simple” deterministic epidemic model is built on the following strongly simplifying assumptions:

(i) At any time point an individual in the population under consideration is either susceptible $(S)$, infected and infectious $(I)$ or recovered and immune $(R)$.

(ii) The disease is transmitted by contact between an infected and infectious individual and a susceptible individual. Only susceptible individuals can get infected and, after having been infectious for some time, an individual recovers and becomes completely immune for the remainder of the study period.

(iii) Throughout the whole study period the population under consideration is fixed in size. There are no births, deaths, immmigration or emigration during the whole study period.


# Simple Kermack-McKendrick SIR Model I

(iv) There is no latency for the disease. The disease is transmitted instantaneously upon contact.


# Simple Kermack-McKendrick SIR Model I

(iv) There is no latency for the disease. The disease is transmitted instantaneously upon contact.

(v) All susceptible individuals are equally susceptible and all infected and infectious individuals are equally infectious.


# Simple Kermack-McKendrick SIR Model I

(iv) There is no latency for the disease. The disease is transmitted instantaneously upon contact.

(v) All susceptible individuals are equally susceptible and all infected and infectious individuals are equally infectious.

"Individuals" can be human beings or animals or trees ($\rightsquigarrow$ forest fire) - but also non-biological entities, such as computers ($\rightsquigarrow$ computer viruses), or financial institutions ($\rightsquigarrow$ infectious defaults).


# Simple Kermack-McKendrick SIR Model I

(iv) There is no latency for the disease. The disease is transmitted instantaneously upon contact.

(v) All susceptible individuals are equally susceptible and all infected and infectious individuals are equally infectious.

"Individuals" can be human beings or animals or trees ($\rightsquigarrow$ forest fire) - but also non-biological entities, such as computers ($\rightsquigarrow$ computer viruses), or financial institutions ($\rightsquigarrow$ infectious defaults).

These assumptions reflect the situation for many diseases, such as measles or influenza, and would seem to be reasonable for computers whose anti-virus software has been updated to recognise the virus.


# Simple Kermack-McKendrick SIR Model II

Consider the evolution of an epidemic in a population of total size $N$. The total population is divided into three classes ("compartments"):


# Simple Kermack-McKendrick SIR Model II

Consider the evolution of an epidemic in a population of total size $N$. The total population is divided into three classes (“compartments”):

$(S)$ the class of susceptibles, i. e. those individuals capable of contracting the disease and becoming themselves infectives;


# Simple Kermack-McKendrick SIR Model II

Consider the evolution of an epidemic in a population of total size $N$. The total population is divided into three classes ("compartments"):

($S$) the class of susceptibles, i. e. those individuals capable of contracting the disease and becoming themselves infectives;

($I$) the class of infectives, i.e. the infected individuals capable of transmitting the disease to susceptibles;


# Simple Kermack-McKendrick SIR Model II

Consider the evolution of an epidemic in a population of total size $N$. The total population is divided into three classes ("compartments"):

- $(S)$ the class of susceptibles, i. e. those individuals capable of contracting the disease and becoming themselves infectives;
- $(I)$ the class of infectives, i.e. the infected individuals capable of transmitting the disease to susceptibles;
- $(R)$ the class of recovered individuals, i.e. those individuals which, having contracted the disease, have recovered, and hence cannot further transmit the disease.


# Simple Kermack-McKendrick SIR Model II

Consider the evolution of an epidemic in a population of total size $N$. The total population is divided into three classes ("compartments"):

($S$) the class of susceptibles, i. e. those individuals capable of contracting the disease and becoming themselves infectives;

($I$) the class of infectives, i.e. the infected individuals capable of transmitting the disease to susceptibles;

($R$) the class of recovered individuals, i.e. those individuals which, having contracted the disease, have recovered, and hence cannot further transmit the disease.

An immediate implication of the assumptions is that individuals can only make two moves: from $S$ to $I$ and from $I$ to $R$. For this reason the model is said to be an SIR epidemic model.


# Simple Kermack-McKendrick SIR Model II

Moreover, in this model $N = S + I + R$ does not change over time (by assumption!).


# Simple Kermack-McKendrick SIR Model II

Moreover, in this model $N = S + I + R$ does not change over time (by assumption!). The independent variable in the simple SIR Model is the time $t$, and the rates of transfer between compartments are mathematically expressed as derivatives with respect to time of the numbers $I$ and $S$ (each one viewed as a differentiable function of $t$).


# Simple Kermack-McKendrick SIR Model II

Moreover, in this model $N = S + I + R$ does not change over time (by assumption!). The independent variable in the simple SIR Model is the time $t$, and the rates of transfer between compartments are mathematically expressed as derivatives with respect to time of the numbers $I$ and $S$ (each one viewed as a differentiable function of $t$). Hence, the simple SIR model is given by a system of 2 coupled non-linear ODEs:

$$\begin{aligned}
\dot{S} := \frac{dS}{dt} &= f_1(S, I) := -\beta S I \\
\dot{I} := \frac{dI}{dt} &= f_2(S, I) := \beta S I - \gamma I = I(\beta S - \gamma) ,
\end{aligned}$$

where $\beta, \gamma > 0$ are positive constants, $S(t) \geq 0$ and $I(t) \geq 0$ for all $t \geq 0$.


# Simple Kermack-McKendrick SIR Model II

$\beta$ is called infection or transmission rate, and $\gamma$ is known as recovery rate.


# Simple Kermack-McKendrick SIR Model II

$\beta$ is called infection or transmission rate, and $\gamma$ is known as recovery rate.

By potentially substituting $\beta$ through $\beta N$ we may assume WLOG that $S + I + R = 1$. These ODEs, together with the initial conditions $I(0) := I_0$ and $S(0) := S_0 := 1 - I_0$ for some fixed $0 < I_0 < 1$ define the model.


# Simple Kermack-McKendrick SIR Model III

In general, there is no analytic solution of this ODE system available. To approximate the solution(s) of the ODE system we have to rely on numerical methods (e. g., Runge-Kutta methods).


# Simple Kermack-McKendrick SIR Model III

In general, there is no analytic solution of this ODE system available. To approximate the solution(s) of the ODE system we have to rely on numerical methods (e. g., Runge-Kutta methods).

## Remark (Lotka-Volterra)

*Coupled non-linear ODE systems of this type are very similar to the Lotka-Volterra equations, also known as the predator-prey equations, given by*

$$\begin{aligned}\dot{S} &= \alpha S - \beta S I = S(\alpha - \beta I)\\ \dot{I} &= \gamma S I - \delta I = I(\gamma S - \delta)\ ,\end{aligned}$$

*where* $\alpha, \beta, \gamma, \delta > 0$ *and* $S, I$ *are defined on* $[0, \infty)$.


# Deterministic generalisations of simple SIR

## Remark (Some deterministic generalisations of the simple SIR model)

*One can add an inflow of newborns into the class $S$ of susceptibles to the simple SIR model, at rate $\mu N$, and - the births balancing - deaths in the classes at rates $-\mu S$, $-\mu I$, $-\mu R$ respectively, for some additional parameter $\mu > 0$. These models are known as classic endemic models.*


# Deterministic generalisations of simple SIR

## Remark (Some deterministic generalisations of the simple SIR model)

*One can add an inflow of newborns into the class $S$ of susceptibles to the simple SIR model, at rate $\mu N$, and - the births balancing - deaths in the classes at rates $-\mu S$, $-\mu I$, $-\mu R$ respectively, for some additional parameter $\mu > 0$. These models are known as classic endemic models. Further slightly more general deterministic models having no immunity (individuals that recover become susceptible immediately) and possibly including age structure and spatial diffusion are called SIS-models (Susceptible - Infected - Susceptible).*


# Deterministic generalisations of simple SIR

## Remark (Some deterministic generalisations of the simple SIR model)

*One can add an inflow of newborns into the class $S$ of susceptibles to the simple SIR model, at rate $\mu N$, and - the births balancing - deaths in the classes at rates $-\mu S$, $-\mu I$, $-\mu R$ respectively, for some additional parameter $\mu > 0$. These models are known as classic endemic models. Further slightly more general deterministic models having no immunity (individuals that recover become susceptible immediately) and possibly including age structure and spatial diffusion are called SIS-models (Susceptible - Infected - Susceptible). Models having a latent state when infected, before becoming infectious, are often called SEIR models ("E" stands for "exposed but not infectious").*


# Deterministic generalisations of simple SIR

## Remark (Some deterministic generalisations of the simple SIR model)

*One can add an inflow of newborns into the class $S$ of susceptibles to the simple SIR model, at rate $\mu N$, and - the births balancing - deaths in the classes at rates $-\mu S$, $-\mu I$, $-\mu R$ respectively, for some additional parameter $\mu > 0$. These models are known as classic endemic models. Further slightly more general deterministic models having no immunity (individuals that recover become susceptible immediately) and possibly including age structure and spatial diffusion are called SIS-models (Susceptible - Infected - Susceptible). Models having a latent state when infected, before becoming infectious, are often called SEIR models ("E" stands for "exposed but not infectious"). Models where immunity decreases after some time are called SIRS models.*


# The basic reproduction ratio

What can we already recognise from the structure of the ODE system itself, without having to know any of its (approximate) solutions?


# The basic reproduction ratio

What can we already recognise from the structure of the ODE system itself, without having to know any of its (approximate) solutions?

## Observation

*Since* $\dot{S} = -\beta\, S\, I$ *on* $(0, \infty)$ *and* $S, I : [0, \infty) \longrightarrow (0, \infty)$ *it follows that* $\dot{S} < 0$ *on* $(0, \infty)$. *Hence, in any case* $S$ *is strictly decreasing on* $(0, \infty)$.


# The basic reproduction ratio

What can we already recognise from the structure of the ODE system itself, without having to know any of its (approximate) solutions?

## Observation

*Since $\dot{S} = -\beta\, S\, I$ on $(0, \infty)$ and $S, I : [0, \infty) \longrightarrow (0, \infty)$ it follows that $\dot{S} < 0$ on $(0, \infty)$. Hence, in any case $S$ is strictly decreasing on $(0, \infty)$. Similarly, since $\dot{I} = (\beta\, S\ - \gamma)\ I$ on $(0, \infty)$ it follows that $I$ is strictly increasing on the set $\{t : t \geq 0 \text{ and } \beta S(t) - \gamma > 0\}$.*


# The basic reproduction ratio

What can we already recognise from the structure of the ODE system itself, without having to know any of its (approximate) solutions?

## Observation

*Since $\dot{S} = -\beta\, S\, I$ on $(0, \infty)$ and $S, I : [0, \infty) \longrightarrow (0, \infty)$ it follows that $\dot{S} < 0$ on $(0, \infty)$. Hence, in any case $S$ is strictly decreasing on $(0, \infty)$. Similarly, since $\dot{I} = (\beta\, S\ - \gamma)\ I$ on $(0, \infty)$ it follows that $I$ is strictly increasing on the set $\{t : t \geq 0$ and $\beta S(t) - \gamma > 0\}$. Moreover, $\frac{d}{dt}\,(S + I) = -\gamma\, I < 0$ on $(0, \infty)$, implying that - in any case - also the non-negative function $S + I$ is strictly decreasing on $(0, \infty)$.*

This leads us to a key parameter in epidemiology:


# The basic reproduction ratio

What can we already recognise from the structure of the ODE system itself, without having to know any of its (approximate) solutions?

## Observation

*Since $\dot{S} = -\beta\, S\, I$ on $(0, \infty)$ and $S, I : [0, \infty) \longrightarrow (0, \infty)$ it follows that $\dot{S} < 0$ on $(0, \infty)$. Hence, in any case $S$ is strictly decreasing on $(0, \infty)$. Similarly, since $\dot{I} = (\beta\, S\, - \gamma)\ I$ on $(0, \infty)$ it follows that $I$ is strictly increasing on the set $\{t : t \geq 0$ and $\beta S(t) - \gamma > 0\}$. Moreover, $\frac{d}{dt}(S + I) = -\gamma\, I < 0$ on $(0, \infty)$, implying that - in any case - also the non-negative function $S + I$ is strictly decreasing on $(0, \infty)$.*

This leads us to a key parameter in epidemiology:

## Definition

$R_0 := \frac{\beta}{\gamma} > 0$ is called basic reproduction ratio.


# Discussion on the role of $R_0$ I

From ODE lectures we know that in the case of the SIR model the following result holds:


# Discussion on the role of $R_0$ I

From ODE lectures we know that in the case of the SIR model the following result holds:

## Proposition

*The initial value problem*

$$\dot{I} = (\beta\, S(t) - \gamma)\, I, I(0) = I_0 \qquad (I_0 \in (0,1])$$

*has an* *uniquely determined positive* *solution on* $[0,\infty)$. *This solution is given by*

$$I(t) := I_0 \,\exp\left(\beta \int_0^t S(u)\,du\right) \exp\left(-\gamma\, t\right) \overset{(!)}{=} I_0 \left( \exp\left( R_0 \left( S(\tau) - \frac{1}{R_0} \right)\right)\right)^{\gamma\, t} \quad (t \geq 0)$$

*for some* $\tau \in (0,t)$.


# Discussion on the role of $R_0$ I

From ODE lectures we know that in the case of the SIR model the following result holds:

## Proposition

*The initial value problem*

$$\dot{I} = (\beta\, S(t) - \gamma)\; I, I(0) = I_0 \qquad (I_0 \in (0,1])$$

*has an uniquely determined positive solution on* $[0,\infty)$*. This solution is given by*

$$I(t) := I_0\, \exp\left(\beta \int_0^t S(u)\,du\right) \exp\left(-\gamma\, t\right) \overset{(!)}{=} I_0 \left(\exp\left(R_0\left(S(\tau) - \frac{1}{R_0}\right)\right)\right)^{\gamma\, t} \quad (t \geq 0)$$

*for some* $\tau \in (0,t)$*.*

Similarly, we can see that also $S(t) > 0$ for all $t \geq 0$ (since $S_0 > 0$ by assumption).


# Discussion on the role of $R_0$ I

From ODE lectures we know that in the case of the SIR model the following result holds:

## Proposition

*The initial value problem*

$$\dot{I} = (\beta\, S(t) - \gamma)\; I, I(0) = I_0 \qquad (I_0 \in (0,1])$$

*has an* *uniquely determined positive* *solution on* $[0,\infty)$. *This solution is given by*

$$I(t) := I_0 \, \exp\left(\beta \int_0^t S(u)\, du\right) \exp\left(-\gamma\, t\right) \overset{(!)}{=} I_0 \left( \exp\left( R_0 \left( S(\tau) - \frac{1}{R_0} \right)\right)\right)^{\gamma\, t} \quad (t \geq 0)$$

*for some* $\tau \in (0,t)$.

Similarly, we can see that also $S(t) > 0$ for all $t \geq 0$ (since $S_0 > 0$ by assumption). Note that *in general* we don't know whether this unique solution $I$ is bounded from above !


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \leq I_0$ *for all* $t \geq 0$


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \leq I_0$ *for all* $t \geq 0$ *(⇝“no large one-time outbreak of the disease”).*


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \le I_0$ *for all* $t \ge 0$ *($\leadsto$"no large one-time outbreak of the disease").*

(ii) $S_0 \le \frac{1}{R_0}$.


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \leq I_0$ *for all* $t \geq 0$ *(⇝"no large one-time outbreak of the disease").*

(ii) $S_0 \leq \frac{1}{R_0}$.

(ii') $I_0 \geq 1 - \frac{1}{R_0}$ *(⇝ meaning!).*


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \leq I_0$ *for all* $t \geq 0$ *($\leadsto$"no large one-time outbreak of the disease").*

(ii) $S_0 \leq \frac{1}{R_0}$.

(ii') $I_0 \geq 1 - \frac{1}{R_0}$ *($\leadsto$ meaning!).*

(iii) *The function $I$ is strictly decreasing on $(0, \infty)$ with limit* $I_\infty := \lim_{t \to \infty} I(t) = 0$.


# Discussion on the role of $R_0$ II

However, $R_0 = \frac{\beta}{\gamma}$ is the key ingredient here, since

## Corollary

*The following statements are equivalent:*

(i) $I(t) \leq I_0$ *for all* $t \geq 0$ *(⇝"no large one-time outbreak of the disease").*

(ii) $S_0 \leq \frac{1}{R_0}$.

(ii') $I_0 \geq 1 - \frac{1}{R_0}$ *(⇝ meaning!).*

(iii) *The function* $I$ *is strictly decreasing on* $(0, \infty)$ *with limit* $I_\infty := \lim_{t\to\infty} I(t) = 0$.

## Proof.

$\underline{(i) \Rightarrow (ii)}$: Assume that $I(t) \leq I_0$ for all $t \geq 0$. Then $\exp\left(\beta \int_0^t S(u)\,du - \gamma\, t\right) \leq 1$ for all $t \geq 0$. Hence, $\frac{1}{t}\int_0^t S(u)\,du \leq \frac{1}{R_0}$ for all $t \geq 0$. Consequently, $S_0 = S(0) = \lim_{t\to 0} \frac{1}{t}\int_0^t S(u)\,du \leq \frac{1}{R_0}$.


# Discussion on the role of $R_0$ II

## Proof ctd.

$\underline{(ii) \Leftrightarrow (ii')}$: algebraically, this is just the initial condition $S_0 = 1 - I_0$. $\underline{(i) \Rightarrow (iii)}$: Let $S_0 \leq \frac{1}{R_0}$. Since $\dot{S} = -\beta\, I\, S$ on $(0, \infty)$ and both, $I$ and $S$ are positive it follows that $\dot{S} < 0$ on $(0, \infty)$, implying that $S$ is a strictly decreasing function on $(0, \infty)$. Hence, $S(t) < S(0) \leq \frac{1}{R_0}$ for all $t > 0$. In particular, $\beta\, S - \gamma < 0$ on $[0, \infty)$. Consequently, $\dot{I} = (\beta\, S - \gamma)\; I < 0$ on $(0, \infty)$, implying that $I$ is strictly decreasing on $(0, \infty)$.


# Discussion on the role of $R_0$ II

## Proof ctd.

$\underline{(ii) \Leftrightarrow (ii')}$: algebraically, this is just the initial condition $S_0 = 1 - I_0$. $\underline{(i) \Rightarrow (iii)}$: Let $S_0 \leq \frac{1}{R_0}$. Since $\dot{S} = -\beta\, I\, S$ on $(0, \infty)$ and both, $I$ and $S$ are positive it follows that $\dot{S} < 0$ on $(0, \infty)$, implying that $S$ is a strictly decreasing function on $(0, \infty)$. Hence, $S(t) < S(0) \leq \frac{1}{R_0}$ for all $t > 0$. In particular, $\beta\, S - \gamma < 0$ on $[0, \infty)$. Consequently, $\dot{I} = (\beta\, S - \gamma)\; I < 0$ on $(0, \infty)$, implying that $I$ is strictly decreasing on $(0, \infty)$. Since $\beta\, S_0 - \gamma < 0$ (by assumption) it follows that

$$\exp\left(\beta \int_0^t S(u)\, du\right) \exp\left(-\gamma\, t\right) \leq \exp\left(\beta\, S(0)\, t\right) \exp\left(-\gamma\, t\right) = \exp\left(-\alpha\, t\right),$$

where $\alpha := -\left(\beta\, S_0 - \gamma\right) > 0$. Hence, $I(t)$ decreases to 0 if $t \to \infty$.


# Discussion on the role of $R_0$ II

## Proof ctd.

$\underline{(ii) \Leftrightarrow (ii')}$: algebraically, this is just the initial condition $S_0 = 1 - I_0$. $\underline{(i) \Rightarrow (iii)}$: Let $S_0 \leq \frac{1}{R_0}$. Since $\dot{S} = -\beta\, I\, S$ on $(0, \infty)$ and both, $I$ and $S$ are positive it follows that $\dot{S} < 0$ on $(0, \infty)$, implying that $S$ is a strictly decreasing function on $(0, \infty)$. Hence, $S(t) < S(0) \leq \frac{1}{R_0}$ for all $t > 0$. In particular, $\beta\, S - \gamma < 0$ on $[0, \infty)$. Consequently, $\dot{I} = (\beta\, S - \gamma)\, I < 0$ on $(0, \infty)$, implying that $I$ is strictly decreasing on $(0, \infty)$. Since $\beta\, S_0 - \gamma < 0$ (by assumption) it follows that

$$\exp\left(\beta \int_0^t S(u)\, du\right) \exp\left(-\gamma\, t\right) \leq \exp\left(\beta\, S(0)\, t\right) \exp\left(-\gamma\, t\right) = \exp\left(-\alpha\, t\right),$$

where $\alpha := -\left(\beta\, S_0 - \gamma\right) > 0$. Hence, $I(t)$ decreases to 0 if $t \to \infty$.

$\underline{(iii) \Rightarrow (i)}$: trivial. $\square$


# Discussion on the role of $R_0$ III

## Observation (Shape of the function $I$ if $R_0 > 1$)

*Let $R_0 > 1$ and $0 < t^*$ such that $S(t^*) = \frac{1}{R_0}$. Then $I$ is strictly increasing on $(0, t^*)$. $I$ attains its single maximum at $t^*$. $I$ is strictly decreasing on $(t^*, \infty)$.*


# Discussion on the role of $R_0$ III

## Observation (Shape of the function $I$ if $R_0 > 1$)

*Let $R_0 > 1$ and $0 < t^*$ such that $S(t^*) = \frac{1}{R_0}$. Then $I$ is strictly increasing on $(0, t^*)$. $I$ attains its single maximum at $t^*$. $I$ is strictly decreasing on $(t^*, \infty)$.*

## Proof.

Elementary differentiation and a bit of elementary algebra induced by the structure of the SIR ODE system shows us that $\dot{I}(t) = 0$ if and only if $t = t^*$ and

$$\ddot{I} \overset{\checkmark}{=} I\ (\beta S - \gamma)^2 - \beta^2 I^2 S - \gamma I$$

on $(0, \infty)$. Consequently, at $t^*$ $I$ attains its single (and hence global) maximum.

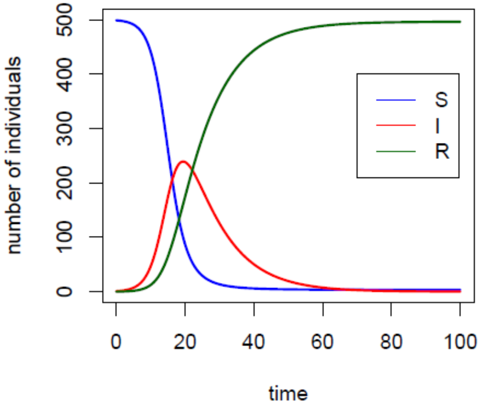

# Discussion on the role of $R_0$ III

Proof ctd.

Let $0 < t < t^*$. Then $\frac{1}{R_0} = S(t^*) < S(t)$, implying that in fact $I$ is strictly increasing on $(0, t^*)$. Now let $t^* < t$. Then $S(t) < S(t^*) = \frac{1}{R_0}$. Hence, $I$ is strictly decreasing on $(t^*, \infty)$. $\square$


# Shape of the functions $S, I$ and $R$ in the case $R_0 > 1$

Here, $N := 500$ and $S, I, R : [0, \infty) \longrightarrow (0, N]$ (not the percentages!).




# Discussion on the role of $R_0$ IV

Even if the basic reproduction ratio $R_0$ is not known to us the number of susceptibles today (i.e., $S_0$) and the percentage of remaining susceptibles $S_\infty := \lim_{t\to\infty} S(t) \leq 1$ when the epidemic is over already allows us to retrieve $R_0$ at least if $S_0 \leq \frac{1}{R_0}$, since:


# Discussion on the role of $R_0$ IV

Even if the basic reproduction ratio $R_0$ is not known to us the number of susceptibles today (i.e., $S_0$) and the percentage of remaining susceptibles $S_\infty := \lim_{t\to\infty} S(t) \leq 1$ when the epidemic is over already allows us to retrieve $R_0$ at least if $S_0 \leq \frac{1}{R_0}$, since:

## Proposition

*Let* $S_0 \leq \frac{1}{R_0}$ *and* $S_0 + I_0 = 1$. *Then*

$$\ln\left(\frac{S_0}{S_\infty}\right) = R_0\left(1 - S_\infty\right) .$$


# Discussion on the role of $R_0$ IV

Proof.

Since $\dot{S}(t) = -\beta\, S(t) I(t)$ for all $t \in (0, \infty)$ and $S(0) = S_0 = 1 - I_0$ we have

$$S(t) = S_0 \exp\left(-\beta \int_0^t I(u)\, du\right) \qquad (t \geq 0)\,.$$

Hence,

$$S(t) \overset{(!)}{=} S_0 \exp\left(R_0 \int_0^t \left(-\gamma I(u)\right)\, du\right)$$


# Discussion on the role of $R_0$ IV

## Proof.

Since $\dot{S}(t) = -\beta\, S(t)I(t)$ for all $t \in (0, \infty)$ and $S(0) = S_0 = 1 - I_0$ we have

$$S(t) = S_0 \exp\left(-\beta \int_0^t I(u)\, du\right) \qquad (t \geq 0)\,.$$

Hence,

$$\begin{aligned} S(t) \quad &\overset{(!)}{=} \quad S_0 \exp\left(R_0 \int_0^t \left(-\gamma I(u)\right)\, du\right) \\ &= \quad S_0 \exp\left(R_0 \left(I(t) + S(t) - 1\right)\right) \quad \text{(why?)}\,. \end{aligned}$$


# Discussion on the role of $R_0$ IV

## Proof ctd.

Equivalently written:

$$\ln\left(\frac{S(0)}{S(t)}\right) = R_0\,(1 - S(t) - I(t))\ .$$

Now we take limits on both sides of the latter equation (as $t \to \infty$). Since $S_0 \leq \frac{1}{R_0}$ by assumption we know that $I_\infty = \lim_{t\to\infty} I(t) = 0$ - and the claim follows. □


1. Revisiting Deterministic Epidemic Models

2. Catching a Glimpse of Stochastic Epidemic Models


# A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time.


# A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time. Think e.g. at nodes in a random graph which change colour according to their state.


# A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time. Think e.g. at nodes in a random graph which change colour according to their state. A bit more formally, fix an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the mappings
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto S_t(\omega)$, respectively
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto I_t(\omega)$, where
$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.


# A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time. Think e.g. at nodes in a random graph which change colour according to their state. A bit more formally, fix an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the mappings
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto S_t(\omega)$, respectively
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto I_t(\omega)$, where
$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

In addition, given the current state of the $S \times S$-valued process $(S_t, I_t)_{t \geq 0}$ at time $t$, we assume that the future state of this process at time $t + \Delta t$, for any $\Delta t > 0$, does not depend on times prior to $t$ (known as Markov property).


# A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time. Think e.g. at nodes in a random graph which change colour according to their state. A bit more formally, fix an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the mappings
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto S_t(\omega)$, respectively
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto I_t(\omega)$, where
$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

In addition, given the current state of the $S \times S$-valued process $(S_t, I_t)_{t \geq 0}$ at time $t$, we assume that the future state of this process at time $t + \Delta t$, for any $\Delta t > 0$, does not depend on times prior to $t$ (known as Markov property). *"The stochastic system has no memory!"*


# A glimpse of the continuous time Markov chain SIR model II

Let $s, i, k, j \in S$ and $t \in [0, \infty)$. Then the associated time-homogeneous transition probabilities

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \mathbb{P}\left((S_{t+\Delta t}, I_{t+\Delta t}) = (s+k, i+j) \;\middle|\; (S_t, I_t) = (s, i)\right)$$

are modelled as

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \begin{cases} \beta s\, i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (-1,1) \\ \gamma i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,-1) \\ 1 - (\beta s\, i - \gamma i)\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,0) \\ o(\Delta t) & \text{else} \end{cases}$$


# A glimpse of the continuous time Markov chain SIR model II

Let $s, i, k, j \in S$ and $t \in [0, \infty)$. Then the associated time-homogeneous transition probabilities

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \mathbb{P}\left((S_{t+\Delta t}, I_{t+\Delta t}) = (s+k, i+j) \;\middle|\; (S_t, I_t) = (s,i)\right)$$

are modelled as

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \begin{cases} \beta s\, i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (-1,1) \\ \gamma i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,-1) \\ 1 - (\beta s\, i - \gamma i)\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,0) \\ o(\Delta t) & \text{else} \end{cases}$$

Unfolding the powerful machinery of Chapman-Kolmogorov equations, respectively time-homogeneous Markov semigroups one can then start to calculate transition probabilities and derived probabilities (think at multiple life insurance mathematics...).


# A glimpse of the SDE SIR model of Allen I

## Coupled SDE SIR Model of Allen

*Let $\beta > 0$, $\gamma > 0$ and $W^{(1)}$ and $W^{(2)}$ be two independent standard Brownian motions.*

$$
\begin{aligned}
dS_t &= -\beta\, S_t\, I_t\, dt - \sqrt{\beta\, S_t\, I_t}\, dW_t^{(1)} \\
dI_t &= \left(\beta\, S_t\, I_t - \gamma\, I_t\right)\, dt + \sqrt{\beta\, S_t\, I_t}\, dW_t^{(1)} - \sqrt{\gamma\, I_t}\, dW_t^{(2)}\,.
\end{aligned}
$$

*Moreover, the initial conditions are given by $I(0) := I_0$, $S(0) := S_0 := 1 - I_0$, $0 < I_0 < 1$ (as in the simple deterministic case).*


# A glimpse of the SDE SIR model of Allen II

One equivalent formulation of the SDE SIR model of Allen is the following one:


# A glimpse of the SDE SIR model of Allen II

One equivalent formulation of the SDE SIR model of Allen is the following one:

## Coupled SDE SIR Model of Allen in Vector Notation

*Let $\beta > 0$, $\gamma > 0$ and $W^{(1)}$ and $W^{(2)}$ be two independent standard Brownian motions.*

$$d\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} -\beta I & 0 \\ \beta I & -\gamma \end{pmatrix} \begin{pmatrix} S \\ I \end{pmatrix} dt + \begin{pmatrix} -\sqrt{\beta S I} & 0 \\ \sqrt{\beta S I} & -\sqrt{\gamma I} \end{pmatrix} d\begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix}$$

*Moreover, the initial conditions are given by* $\begin{pmatrix} S(0) \\ I(0) \end{pmatrix} := \begin{pmatrix} S_0 \\ I_0 \end{pmatrix}$, *where* $S_0 := 1 - I_0$, $0 < I_0 < 1$ *(as in the simple deterministic case).*


# A glimpse of the SDE SIR model of Allen III

## Problem

*Is the SDE SIR model of Allen well-defined ? To answer this non-trivial question we need the whole machinery of (multidimensional) Itô calculus including theory and application of the (vector-valued stochastic) Itô integral ! Actually, SDEs are stochastic integral equations. Brownian motion paths are nowhere differentiable with probability 1!*


# A glimpse of the SDE SIR model of Allen III

## Problem

*Is the SDE SIR model of Allen well-defined ? To answer this non-trivial question we need the whole machinery of (multidimensional) Itô calculus including theory and application of the (vector-valued stochastic) Itô integral ! Actually, SDEs are stochastic integral equations. Brownian motion paths are nowhere differentiable with probability 1! Do data reflect whether this model is useful in practice? Why are two independent standard Brownian motions used?*


# A glimpse of the SDE SIR model of Allen III

## Problem

*Is the SDE SIR model of Allen well-defined ? To answer this non-trivial question we need the whole machinery of (multidimensional) Itô calculus including theory and application of the (vector-valued stochastic) Itô integral ! Actually, SDEs are stochastic integral equations. Brownian motion paths are nowhere differentiable with probability 1! Do data reflect whether this model is useful in practice? Why are two independent standard Brownian motions used? What about the possibility of Poisson jumps? Are the solution processes* $S$ *and* $I$ *still semimartingales?*


# Only a - very - few references

[1] L. J. S. Allen.
An Introduction to Stochastic Epidemic Models.
*In: Brauer F., van den Driessche P., Wu J. (eds) Mathematical Epidemiology. Lecture Notes in Mathematics, vol 1945, Springer, Berlin, Heidelberg (2008)*

[2] L. J. S. Allen.
A primer on stochastic epidemic models: formulation, numerical simulation, and analysis.
*Infectious Disease Modelling 2, 128-142 (2017)*.

[3] E. Beretta and V. Capasso.
On the general structure of epidemic systems.
*Global asymptotic stability. Comp. and Maths. with Appls. 12A, 677-694 (1986)*.


# Only a - very - few references

[4] F. Brauer and C. Castillo-Chavez.
*Mathematical Models in Population Biology and Epidemiology. Second Edition*.
Texts in Applied Mathematics (TAM), vol 40, Springer Berlin Heidelberg (2012).

[5] F. Brauer, P. van den Driessche, J. Wu.
*Mathematical Epidemiology*.
Lecture Notes in Mathematics, vol 1945, Springer Berlin Heidelberg (2008).

[6] T. Britton.
Stochastic epidemic models: a survey.
*Math. Biosci. 225, 24-35 (2010)*.


# Only a - very - few references

[7] H. W. Hethcote.
The Mathematics of Infectious Diseases.
*SIAM Review, Vol. 42, No.* **4***, 599-653 (2000)*.

[8] J. Hofbauer and K. Sigmund.
*The Theory of Evolution and Dynamical Systems*.
Cambridge Univ. Press, Cambridge, U.K. (1988).

[9] W. O. Kermack and A. G. McKendrick.
A contribution to the mathematical theory of epidemics.
*Proc. Royal Soc. Lond. Series A, 115:700-721 (1927)*.


Thank you for your attention!

*Are there any questions, comments or remarks?*