Mathematical Modelling of Infectious Diseases for Public Health

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Public Health England, Porton Down, Salisbury - Discussion Meeting

14.09.2018

Contents

1. Revisiting Deterministic Epidemic Models

2. Catching a Glimpse of Stochastic Epidemic Models

Simple Kermack-McKendrick SIR Model I

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(iii) Throughout the whole study period the population under consideration is fixed in size. There are no births, deaths, immmigration or emigration during the whole study period.

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These assumptions reflect the situation for many diseases, such as measles or influenza, and would seem to be reasonable for computers whose anti-virus software has been updated to recognise the virus.

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An immediate implication of the assumptions is that individuals can only make two moves: from S to I and from I to R. For this reason the model is said to be an SIR epidemic model.

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Moreover, in this model $N = S + I + R$ does not change over time (by assumption!). The independent variable in the simple SIR Model is the time t, and the rates of transfer between compartments are mathematically expressed as derivatives with respect to time of the numbers I and S (each one viewed as a differentiable function of t). Hence, the simple SIR model is given by a system of 2 coupled non-linear ODEs:

$$\begin{aligned} \dot{S} := \frac{dS}{dt} &= f_1(S, I) := -\beta S I \\ \dot{I} := \frac{dI}{dt} &= f_2(S, I) := \beta S I - \gamma I = I(\beta S - \gamma) \,, \end{aligned}$$

where $\beta, \gamma > 0$ are positive constants, $S(t) \geq 0$ and $I(t) \geq 0$ for all $t \geq 0$.

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By potentially substituting β through βN we may assume WLOG that $S + I + R = 1$. These ODEs, together with the initial conditions $I(0) := I_0$ and $S(0) := S_0 := 1 - I_0$ for some fixed $0 < I_0 < 1$ define the model.

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Remark (Lotka-Volterra)

Coupled non-linear ODE systems of this type are very similar to the Lotka-Volterra equations, also known as the predator-prey equations, given by

$$\begin{aligned} \dot{S} &= \alpha S - \beta S I = S(\alpha - \beta I) \\ \dot{I} &= \gamma S I - \delta I = I(\gamma S - \delta) \,, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta > 0$ *and* S, I *are defined on* $[0, \infty)$.

Deterministic generalisations of simple SIR

Remark (Some deterministic generalisations of the simple SIR model)

One can add an inflow of newborns into the class S of susceptibles to the simple SIR model, at rate μN, and - the births balancing - deaths in the classes at rates $-\mu S$, $-\mu I$, $-\mu R$ respectively, for some additional parameter $\mu > 0$. These models are known as classic endemic models.

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The basic reproduction ratio

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Observation

Since $\dot{S} = -\beta\, S\, I$ *on* $(0, \infty)$ *and* $S, I : [0, \infty) \longrightarrow (0, \infty)$ *it follows that* $\dot{S} < 0$ *on* $(0, \infty)$. *Hence, in any case* S *is strictly decreasing on* $(0, \infty)$.

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Definition

$R_0 := \frac{\beta}{\gamma} > 0$ is called basic reproduction ratio.

Discussion on the role of R_0 I

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Proposition

The initial value problem

$$\dot{I} = (\beta\, S(t) - \gamma)\; I, I(0) = I_0 \qquad (I_0 \in (0,1])$$

has an uniquely determined positive *solution on* $[0,\infty)$*. This solution is given by*

$$I(t) := I_0 \,\exp\left(\beta \int_0^t S(u)\, du\right) \exp\left(-\gamma\, t\right) \qquad (t \geq 0)\,.$$

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Similarly, we can see that also $S(t) > 0$ for all $t \geq 0$ (since $S_0 > 0$ by assumption). Note that *in general* we don't know whether this unique solution I is bounded from above !

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Proof.

$\underline{(i) \Rightarrow (ii)}$: Assume that $I(t) \leq I_0$ for all $t \geq 0$. Then $\exp\left(\beta \int_0^t S(u)\,du - \gamma\, t\right) \leq 1$ for all $t \geq 0$. Hence, $\frac{1}{t} \int_0^t S(u)\,du \leq \frac{1}{R_0}$ for all $t \geq 0$. Consequently, $S_0 = S(0) = \lim_{t \to 0} \frac{1}{t} \int_0^t S(u)\,du \leq \frac{1}{R_0}$.

Discussion on the role of R_0 II

Proof ctd.

$\underline{(i) \Rightarrow (iii)}$: Let $S_0 \leq \frac{1}{R_0}$. Since $\dot{S} = -\beta\, I\, S$ on $(0, \infty)$ and both, I and S are positive it follows that $\dot{S} < 0$ on $(0, \infty)$, implying that S is a strictly decreasing function on $(0, \infty)$. Hence, $S(t) < S(0) \leq \frac{1}{R_0}$ for all $t > 0$. In particular, $\beta\, S - \gamma < 0$ on $[0, \infty)$. Consequently, $\dot{I} = (\beta\, S - \gamma)\; I < 0$ on $(0, \infty)$, implying that I is strictly decreasing on $(0, \infty)$.

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$$\exp\left(\beta \int_0^t S(u)\, du\right) \exp\left(-\gamma\, t\right) \leq \exp\left(\beta\, S(0)\, t\right) \exp\left(-\gamma\, t\right) = \exp\left(-\alpha\, t\right),$$

where $\alpha := -\left(\beta\, S_0 - \gamma\right) > 0$. Hence, $I(t)$ decreases to 0 if $t \to \infty$.

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where $\alpha := -\left(\beta\, S_0 - \gamma\right) > 0$. Hence, $I(t)$ decreases to 0 if $t \rightarrow \infty$.

$\underline{(iii) \Rightarrow (i)}$: trivial. □

Discussion on the role of R_0 III

Observation (Shape of the function I if $R_0 > 1$)

Let $R_0 > 1$ and $0 < t^*$ such that $S(t^*) = \frac{1}{R_0}$. Then I is strictly increasing on $(0, t^*)$. I attains its single maximum at t^*. I is strictly decreasing on (t^*, ∞).

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Proof.

Elementary differentiation and a bit of elementary algebra induced by the structure of the SIR ODE system shows us that $\dot{I}(t) = 0$ if and only if $t = t^*$ and

$$\ddot{I} \stackrel{\checkmark}{=} I\ (\beta S - \gamma)^2 - \beta^2 I^2 S - \gamma I$$

on $(0, \infty)$. Consequently, at t^* I attains its single (and hence global) maximum.

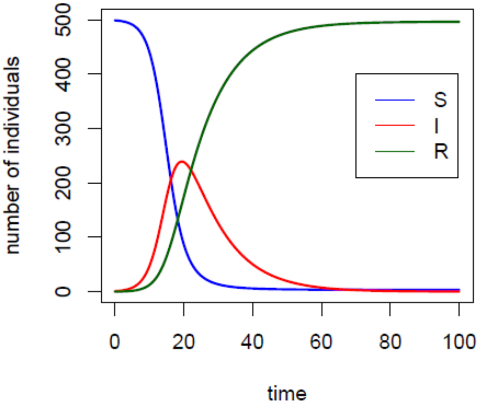
Discussion on the role of R_0 III

Proof ctd.

Let $0 < t < t^*$. Then $\frac{1}{R_0} = S(t^*) < S(t)$, implying that in fact I is strictly increasing on $(0, t^*)$. Now let $t^* < t$. Then $S(t) < S(t^*) = \frac{1}{R_0}$. Hence, I is strictly decreasing on (t^*, ∞). □

Shape of the functions S, I and R in the case $R_0 > 1$

Here, $N := 500$ and $S, I, R : [0, \infty) \longrightarrow (0, N]$ (not the percentages!).



Discussion on the role of R_0 IV

Even if the basic reproduction ratio R_0 is not known to us the number of susceptibles today (i.e., S_0) and the percentage of remaining susceptibles $S_\infty := \lim_{t\to\infty} S(t) \leq 1$ when the epidemic is over already allows us to retrieve R_0 at least if $S_0 \leq \frac{1}{R_0}$, since:

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Proposition

Let $S_0 \leq \frac{1}{R_0}$ *and* $S(0) + I(0) = 1$. *Then*

$$\ln\left(\frac{S_0}{S_\infty}\right) = R_0\left(1 - S_\infty\right) .$$

Discussion on the role of R_0 IV

Proof.

Since $\dot{S}(t) = -\beta\, S(t)I(t)$ for all $t \in (0, \infty)$ and $S(0) = S_0 = 1 - I_0$ we have

$$S(t) = S_0 \exp\left(-\beta \int_0^t I(u)\, du\right) \qquad (t \geq 0)\,.$$

Hence,

$$S(t) \stackrel{(!)}{=} S_0 \exp\left(R_0 \int_0^t \left(-\gamma I(u)\right)\, du\right)$$

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$$S(t) = S_0 \exp\left(-\beta \int_0^t I(u)\, du\right) \qquad (t \geq 0)\,.$$

Hence,

$$\begin{aligned} S(t) &\overset{(!)}{=} S_0 \exp\left(R_0 \int_0^t \left(-\gamma I(u)\right)\, du\right) \\ &= S_0 \exp\left(R_0 \left(I(t) + S(t) - 1\right)\right) \quad \text{(why?)}\,. \end{aligned}$$

Discussion on the role of R_0 IV

Proof ctd.

Equivalently written:

$$\ln\left(\frac{S(0)}{S(t)}\right) = R_0\left(1 - S(t) - I(t)\right) .$$

Now we take limits on both sides of the latter equation (as $t \to \infty$). Since $S_0 \leq \frac{1}{R_0}$ by assumption we know that $I_\infty = \lim_{t\to\infty} I(t) = 0$ - and the claim follows. $\square$


Susceptible (S)
βSI
Infectious (I)
γI
Recovered (R)

A glimpse of the continuous time Markov chain SIR model I

View the class of susceptibles, respectively the class of infected as stochastic random variables, changing randomly in time.

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$\Omega \times [0, \infty) \ni (\omega, t) \mapsto S_t(\omega)$, respectively
$\Omega \times [0, \infty) \ni (\omega, t) \mapsto I_t(\omega)$, where
$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

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$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

In addition, given the current state of the $S \times S$-valued process $(S_t, I_t)_{t \geq 0}$ at time t, we assume that the future state of this process at time $t + \Delta t$, for any $\Delta t > 0$, does not depend on times prior to t (known as Markov property).

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$\Omega \times [0, \infty) \ni (\omega, t) \mapsto I_t(\omega)$, where
$I_t(\omega), S_t(\omega) \in S := \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$ and $t \in [0, \infty)$.

In addition, given the current state of the $S \times S$-valued process $(S_t, I_t)_{t \geq 0}$ at time t, we assume that the future state of this process at time $t + \Delta t$, for any $\Delta t > 0$, does not depend on times prior to t (known as Markov property). *“The stochastic system has no memory!”*

A glimpse of the continuous time Markov chain SIR model II

Let $s, i, k, j \in S$ and $t \in [0, \infty)$. Then the associated time-homogeneous transition probabilities

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \mathbb{P}\left((S_{t+\Delta t}, I_{t+\Delta t}) = (s+k, i+j) \;\middle|\; (S_t, I_t) = (s, i)\right)$$

are modelled as

$$p_{(s,i),(s+k,i+j)}\,(\Delta t) := \begin{cases} \beta s\, i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (-1,1) \\ \gamma i\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,-1) \\ 1 - (\beta s\, i - \gamma i)\, \Delta t + o(\Delta t) & \text{if } (k,j) = (0,0) \\ o(\Delta t) & \text{else} \end{cases}$$

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Unfolding the powerful machinery of Chapman-Kolmogorov equations, respectively time-homogeneous Markov semigroups one can then start to calculate transition probabilities and derived probabilities (think at multiple life insurance mathematics...).

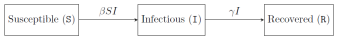
A glimpse of the SDE SIR model of Allen I

Coupled SDE SIR Model of Allen

Let $\beta > 0$, $\gamma > 0$ and $W^{(1)}$ and $W^{(2)}$ be two independent standard Brownian motions.

$$\begin{aligned} dS_t &= -\beta\, S_t\, I_t\, dt - \sqrt{\beta\, S_t\, I_t}\, dW_t^{(1)} \\ dI_t &= (\beta\, S_t\, I_t - \gamma\, I_t)\; dt + \sqrt{\beta\, S_t\, I_t}\, dW_t^{(1)} - \sqrt{\gamma\, I_t}\, dW_t^{(2)}\,. \end{aligned}$$

Moreover, the initial conditions are given by $I(0) := I_0$, $S(0) := S_0 := 1 - I_0$, $0 < I_0 < 1$ (as in the simple deterministic case).



A glimpse of the SDE SIR model of Allen II

One equivalent formulation of the SDE SIR model of Allen is the following one:

A glimpse of the SDE SIR model of Allen II

One equivalent formulation of the SDE SIR model of Allen is the following one:

Coupled SDE SIR Model of Allen in Vector Notation

Let $\beta > 0$, $\gamma > 0$ and $W^{(1)}$ and $W^{(2)}$ be two independent standard Brownian motions.

$$d\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} -\beta I & 0 \\ \beta I & -\gamma \end{pmatrix}\begin{pmatrix} S \\ I \end{pmatrix} dt + \begin{pmatrix} -\sqrt{\beta S I} & 0 \\ \sqrt{\beta S I} & -\sqrt{\gamma I} \end{pmatrix} d\begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix}$$

Moreover, the initial conditions are given by $\begin{pmatrix} S(0) \\ I(0) \end{pmatrix} := \begin{pmatrix} S_0 \\ I_0 \end{pmatrix}$*,*
where $S_0 := 1 - I_0$, $0 < I_0 < 1$ (as in the simple deterministic case).

A glimpse of the SDE SIR model of Allen III

Problem

Is the SDE SIR model of Allen well-defined ? To answer this non-trivial question we need the whole machinery of (multidimensional) Itô calculus including theory and application of the (vector-valued stochastic) Itô integral ! Actually, SDEs are stochastic integral equations. Brownian motion paths are nowhere differentiable with probability 1!

A glimpse of the SDE SIR model of Allen III

Problem

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A glimpse of the SDE SIR model of Allen III

Problem

Is the SDE SIR model of Allen well-defined ? To answer this non-trivial question we need the whole machinery of (multidimensional) Itô calculus including theory and application of the (vector-valued stochastic) Itô integral ! Actually, SDEs are stochastic integral equations. Brownian motion paths are nowhere differentiable with probability 1! Do data reflect whether this model is useful in practice? Why are two independent standard Brownian motions used? What about the possibility of Poisson jumps? Are the solution processes S *and* I *still semimartingales?*

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Thank you for your attention!

Are there any questions, comments or remarks?