A statistical interpretation of Grothendieck's inequality and its relation to the size of non-locality of quantum mechanics

Frank Oertel

CPNSS, LSE

26 February 2018

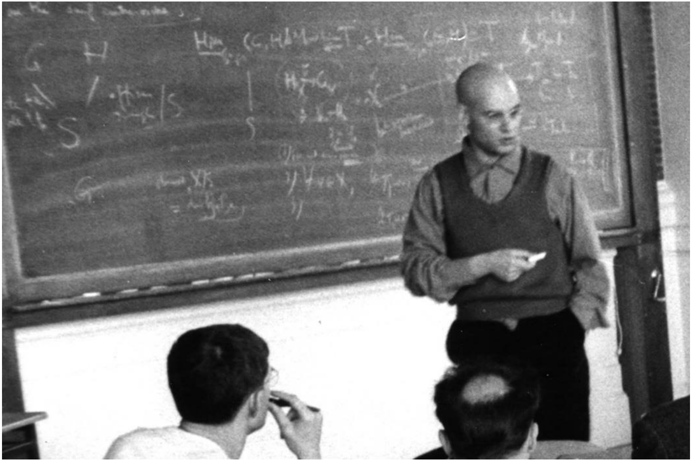
Sigma Club

Contents

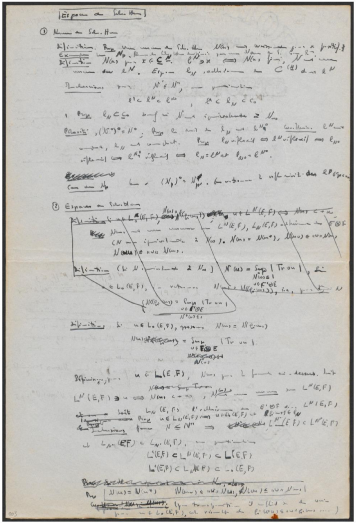
1. A very short glimpse at A. Grothendieck's work in functional analysis
2. Grothendieck's inequality in matrix formulation
3. Grothendieck's inequality rewritten
4. Grothendieck's inequality and correlation matrices
5. Towards a calculation of Grothendieck's constant $K_G^{\mathbb{R}}$
6. Grothendieck's inequality and its relation to non-locality in quantum mechanics

A portrait of A. Grothendieck

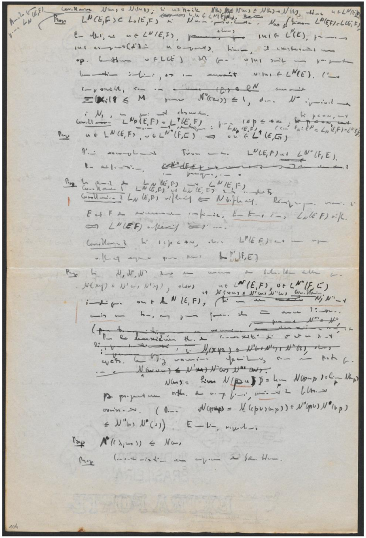
A. Grothendieck lecturing at IHES (1958-1970)



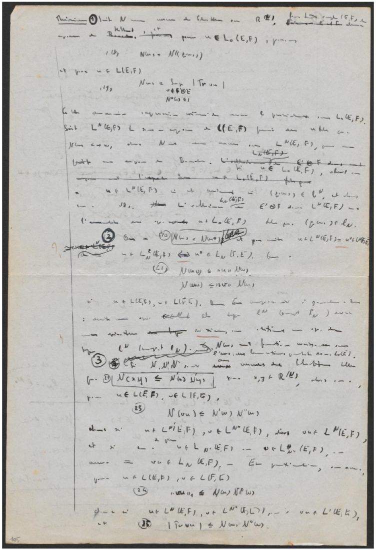
Excerpt from A. Grothendieck's handwritten lecture notes I



Excerpt from A. Grothendieck's handwritten lecture notes II



Excerpt from A. Grothendieck's handwritten lecture notes III



Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $m, n \in \mathbb{N}$. *Let . Then there exists a universal constant* $K > 0$ *- not depending on* m *and* n *- such that for all matrices* $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{F})$*, all* $\mathbb{F}$*-Hilbert spaces* H*, all unit vectors* $u_1, \ldots, u_m, v_1, \ldots, v_n \in H$ *the following inequality is satisfied:*

$$\Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K \, \max \Big\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| : p_i, q_j \in \{-1, 1\} \Big\} \, .$$

Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Let . Then there exists a universal constant $K > 0$ - not depending on m and n - such that for all matrices $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$, all $\mathbb{F}$-Hilbert spaces H, all unit vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in H$ the following inequality is satisfied:

$$\Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K \max \Big\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| : p_i, q_j \in \{-1, 1\} \Big\}.$$

The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant.

Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Let . Then there exists a universal constant $K > 0$ - not depending on m and n - such that for all matrices $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$, all $\mathbb{F}$-Hilbert spaces H, all unit vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in H$ the following inequality is satisfied:

$$\Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K \max \Big\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| : p_i, q_j \in \{-1, 1\} \Big\}.$$

The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant. Computing the exact numerical value of this constant is an open problem (unsolved since 1953)!

Grothendieck's inequality in matrix form II

An easy exercise shows that GT is equivalent to

Grothendieck's inequality in matrix form II

An easy exercise shows that GT is equivalent to

Theorem

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Let $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{F})$. Then there exists a universal constant $K > 0$ such that the following property is satisfied:

Grothendieck's inequality in matrix form II

An easy exercise shows that GT is equivalent to

Theorem

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Let $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{F})$. Then there exists a universal *constant $K > 0$ such that the following property is satisfied:*

If

$$\max \left\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| : p_i, q_j \in \{-1, 1\} \right\} \leq 1$$

then

$$\Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K.$$

for all Hilbert spaces over $\mathbb{F}$ and all unit vectors $u_1, \ldots, u_m$, $v_1, \ldots, v_n \in H$.

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* H *be an arbitrary Hilbert space over* $\mathbb{F}$*. Let* $n \in \mathbb{N}$ *and* $A = \big(a_{ij}\big) \in \mathbb{M}(n \times n; \mathbb{F})$ *be Hermitian and positive semidefinite. Then*

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and H be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$ and $A = \left(a_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{F})$ be Hermitian and positive semidefinite. Then

$$K_G^{\mathbb{R}} = \frac{\pi}{2}$$

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* H *be an arbitrary Hilbert space over* $\mathbb{F}$*. Let* $n \in \mathbb{N}$ *and* $A = \left(a_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{F})$ *be Hermitian and positive semidefinite. Then*

$$K_G^{\mathbb{R}} = \frac{\pi}{2} \quad \textit{and}$$

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* H *be an arbitrary Hilbert space over* $\mathbb{F}$*. Let* $n \in \mathbb{N}$ *and* $A = \big(a_{ij}\big) \in \mathbb{M}(n \times n; \mathbb{F})$ *be Hermitian and positive semidefinite. Then*

$$K_G^{\mathbb{R}} = \frac{\pi}{2} \quad \text{and} \quad K_G^{\mathbb{C}} = \frac{4}{\pi}\,.$$

Grothendieck's inequality in matrix form III

Thanks to a strong use of vector measures, representing kernel Hilbert spaces (RKHS's) and Hilbert space-valued stochastic processes, all applied by members of probability schools we may list a sharp value for $K_G^{\mathbb{F}}$ in the following particular case:

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and H be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$ and $A = \big(a_{ij}\big) \in \mathbb{M}(n \times n; \mathbb{F})$ be Hermitian and positive semidefinite. Then

$$K_G^{\mathbb{R}} = \frac{\pi}{2} \quad \text{and} \quad K_G^{\mathbb{C}} = \frac{4}{\pi} \,.$$

From now on are going to consider the real case (i. e., $\mathbb{F} = \mathbb{R}$) only. Nevertheless, we allow an unrestricted use of all matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ for any $m, n \in \mathbb{N}$.

Grothendieck's inequality in matrix form IV

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

Grothendieck's inequality in matrix form IV

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

$$1,676 < K_G^{\mathbb{R}} < \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1,782\,.$$

Grothendieck's inequality in matrix form IV

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

$$1,676 < K_G^{\mathbb{R}} < \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1,782\,.$$

Screening these numbers we might be tempted to guess the following

Grothendieck's inequality in matrix form IV

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily due to R. E. Rietz (1974), J. L. Krivine (1977), and most recently, M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (4-author paper from 2011, available on the arXiv)*:

$$1,676 < K_G^{\mathbb{R}} < \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1,782\,.$$

Screening these numbers we might be tempted to guess the following

Conjecture

Is $K_G^{\mathbb{R}} = \sqrt{\pi} \approx 1,772$*?*

1. A very short glimpse at A. Grothendieck's work in functional analysis
2. Grothendieck's inequality in matrix formulation
3. Grothendieck's inequality rewritten
4. Grothendieck's inequality and correlation matrices
5. Towards a calculation of Grothendieck's constant $K_G^{\mathbb{R}}$
6. Grothendieck's inequality and its relation to non-locality in quantum mechanics

Grothendieck's inequality rewritten I

By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear transformations of correlation matrices.

Grothendieck's inequality rewritten I

By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear transformations of correlation matrices.

We will sketch this approach which might lead to a constructive improvement of Krivine's upper bound $\frac{\pi}{2\ln(1+\sqrt{2})}$. At least it also can be reproduced in this approach.

Grothendieck's inequality rewritten I

By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear transformations of correlation matrices.

We will sketch this approach which might lead to a constructive improvement of Krivine's upper bound $\frac{\pi}{2\ln(1+\sqrt{2})}$. At least it also can be reproduced in this approach.

When the context is clear we suppress the Hilbert space symbol "H" and use the notation "$\langle\cdot,\cdot\rangle$" instead of "$\langle\cdot,\cdot\rangle_H$".

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H =$$

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \mathrm{tr}\big(A^\top \, \Gamma_H(u, v)\big) =$$

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \operatorname{tr}\big(A^\top \, \Gamma_H(u, v)\big) = \langle A, \Gamma_H(u, v) \rangle,$$

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \mathrm{tr}\big(A^\top \Gamma_H(u, v)\big) = \langle A, \Gamma_H(u, v) \rangle,$$

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where

$$\Gamma_H(u, v) := \begin{pmatrix} \langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \ldots & \langle u_1, v_n \rangle_H \\ \langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \ldots & \langle u_2, v_n \rangle_H \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \ldots & \langle u_m, v_n \rangle_H \end{pmatrix}.$$

Grothendieck's inequality rewritten: Bell is lurking

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Grothendieck's inequality rewritten: Bell is lurking

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \mathrm{tr}\left(A^\top \Gamma_{\mathbb{R}}(p, q)\right) = \langle A, \Gamma_{\mathbb{R}}(p, q)\rangle,$$

where now

Grothendieck's inequality rewritten: Bell is lurking

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \mathrm{tr}\big(A^\top \, \Gamma_{\mathbb{R}}(p, q)\big) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,$$

where now

$$\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_n \\ p_2 q_1 & p_2 q_2 & \cdots & p_2 q_n \\ \vdots & \vdots & \vdots & \vdots \\ p_m q_1 & p_m q_2 & \cdots & p_m q_n \end{pmatrix}.$$

Grothendieck's inequality rewritten: Bell is lurking

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \operatorname{tr}\left(A^\top \Gamma_{\mathbb{R}}(p, q)\right) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,$$

where now

$$\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix} \langle p_1, q_1 \rangle_{\mathbb{R}} & \langle p_1, q_2 \rangle_{\mathbb{R}} & \ldots & \langle p_1, q_n \rangle_{\mathbb{R}} \\ \langle p_2, q_1 \rangle_{\mathbb{R}} & \langle p_2, q_2 \rangle_{\mathbb{R}} & \ldots & \langle p_2, q_n \rangle_{\mathbb{R}} \\ \vdots & \vdots & \vdots & \vdots \\ \langle p_m, q_1 \rangle_{\mathbb{R}} & \langle p_m, q_2 \rangle_{\mathbb{R}} & \ldots & \langle p_m, q_n \rangle_{\mathbb{R}} \end{pmatrix}.$$

Grothendieck's inequality rewritten: Bell is lurking

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \operatorname{tr}\big(A^\top \Gamma_{\mathbb{R}}(p, q)\big) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,$$

where now

$$\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix} \pm 1 & \mp 1 & \ldots & \pm 1 \\ \mp 1 & \mp 1 & \ldots & \mp 1 \\ \vdots & \vdots & \vdots & \vdots \\ \pm 1 & \mp 1 & \ldots & \pm 1 \end{pmatrix}.$$

Full matrix representation of the Hilbert space vectors

Pick all $m+n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

Full matrix representation of the Hilbert space vectors

Pick all $m+n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

$$\begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \ldots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \ldots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \ldots & \langle u_m, v_n\rangle \end{pmatrix}$$

Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

$$\begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle \end{pmatrix}$$

Does this matrix look familiar to you?

Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

$$\begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle \end{pmatrix}$$

Does this matrix look familiar to you?

It is a part of something larger...

Full matrix representation of the Hilbert space vectors

Pick all $m+n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

$$\begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \ldots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \ldots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \ldots & \langle u_m, v_n\rangle \end{pmatrix}$$

Does this matrix look familiar to you?

It is a part of something larger...

Namely:

Block matrix representation I

$$\begin{pmatrix} & \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix} \\ \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}^{\mathbf{T}} & \end{pmatrix}$$

Block matrix representation I

$$
\begin{pmatrix}
 & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
 & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle u_1, v_1\rangle & \langle u_2, v_1\rangle & \dots & \langle u_m, v_1\rangle & & & & \\
\langle u_1, v_2\rangle & \langle u_2, v_2\rangle & \dots & \langle u_m, v_2\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle u_1, v_n\rangle & \langle u_2, v_n\rangle & \dots & \langle u_m, v_n\rangle & & & &
\end{pmatrix}
$$

Block matrix representation I

$$
\left(\begin{array}{cccccccc}
 & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
 & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & & & & \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & & & &
\end{array}\right)
$$

Block matrix representation II

$$
\begin{pmatrix}
\langle u_1, u_1\rangle & \langle u_1, u_2\rangle & \dots & \langle u_1, u_m\rangle & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
\langle u_2, u_1\rangle & \langle u_2, u_2\rangle & \dots & \langle u_2, u_m\rangle & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1\rangle & \langle u_m, u_2\rangle & \dots & \langle u_m, u_m\rangle & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & \langle v_n, v_1\rangle & \langle v_m, v_2\rangle & \dots & \langle v_n, v_n\rangle
\end{pmatrix}
$$

Block matrix representation III

$$
\begin{pmatrix}
1 & \langle u_1, u_2\rangle & \dots & \langle u_1, u_m\rangle & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
\langle u_2, u_1\rangle & 1 & \dots & \langle u_2, u_m\rangle & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1\rangle & \langle u_m, u_2\rangle & \dots & 1 & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & 1 & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & \langle v_2, v_1\rangle & 1 & \dots & \langle v_2, v_n\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & \langle v_n, v_1\rangle & \langle v_m, v_2\rangle & \dots & 1
\end{pmatrix}
$$

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$PSD(n;\mathbb{R}) := \{S : S^\top = S \in \mathbb{M}(n \times n;\mathbb{R}), S \text{ is positive semidefinite}\}.$$

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$PSD(n;\mathbb{R}) := \{S : S^\top = S \in \mathbb{M}(n \times n;\mathbb{R}), S \text{ is positive semidefinite}\}.$$

Recall that $PSD(n;\mathbb{R})$ is a closed convex cone.

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$PSD(n;\mathbb{R}) := \{S : S^\top = S \in \mathbb{M}(n \times n;\mathbb{R}), S \text{ is positive semidefinite}\}.$$

Recall that $PSD(n;\mathbb{R})$ is a closed convex cone.

Moreover, we consider the set

$$C(n;\mathbb{R}) := \Big\{S \in PSD(n;\mathbb{R}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\Big\}.$$

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$PSD(n;\mathbb{R}) := \{S : S^\top = S \in \mathbb{M}(n \times n;\mathbb{R}), S \text{ is positive semidefinite}\}.$$

Recall that $PSD(n;\mathbb{R})$ is a closed convex cone.

Moreover, we consider the set

$$C(n;\mathbb{R}) := \left\{S \in PSD(n;\mathbb{R}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\right\}.$$

The following statement (resulting from spectral decomposition/SVD) is of utmost importance:

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$$PSD(n;\mathbb{R}) := \{S : S^\top = S \in \mathbb{M}(n \times n;\mathbb{R}), S \text{ is positive semidefinite}\}.$$

Recall that $PSD(n;\mathbb{R})$ is a closed convex cone.

Moreover, we consider the set

$$C(n;\mathbb{R}) := \left\{S \in PSD(n;\mathbb{R}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\right\}.$$

The following statement (resulting from spectral decomposition/SVD) is of utmost importance:

Theorem (Square roots in $PSD(n;\mathbb{R})$)

Let $n \in \mathbb{N}$ and $S \in \mathbb{M}(n \times n;\mathbb{R})$. TFAE:

(i) $S \in PSD(n;\mathbb{R})$.

(ii) $S = B^2$ *for some* $B \in PSD(n;\mathbb{R})$.

This $B \in PSD(n;\mathbb{R})$ is unique and called the square root of S: $S^{1/2} := B$.

A refresher of a few definitions II

Let $d, k \in \mathbb{N}$ and $(H, \langle\cdot,\cdot\rangle)$ be an arbitrary d-dimensional Hilbert space (i. e, $H = l_2^d$). Let $w_1, w_2, \ldots, w_k \in H$. Put
$w := (w_1, \ldots, w_k)^\top \in H^k$ and
$S := \big(w_1 \,\vert\, w_2 \,\vert\, \ldots \,\vert\, w_k\big) \in \mathbb{M}(d \times k; \mathbb{R})$. The matrix $\Gamma_H(w, w)$
$\in PSD(k; \mathbb{R})$, defined as

$$\Gamma_H(w, w)_{ij} := \langle w_i, w_j\rangle = \big(S^\top S\big)_{ij} \qquad \big(i, j \in [k] := \{1, 2, \ldots, k\}\big)$$

is called Gram matrix of the vectors $w_1, \ldots, w_k \in H$.

A refresher of a few definitions II

Let $d, k \in \mathbb{N}$ and $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary d-dimensional Hilbert space (i. e, $H = l_2^d$). Let $w_1, w_2, \ldots, w_k \in H$. Put
$w := (w_1, \ldots, w_k)^\top \in H^k$ and
$S := \big(w_1 \,\vert\, w_2 \,\vert\, \ldots \,\vert\, w_k\big) \in \mathbb{M}(d \times k; \mathbb{R})$. The matrix $\Gamma_H(w, w)$
$\in PSD(k; \mathbb{R})$, defined as

$$\Gamma_H(w, w)_{ij} := \langle w_i, w_j \rangle = \left(S^\top S\right)_{ij} \qquad \left(i, j \in [k] := \{1, 2, \ldots, k\}\right)$$

is called Gram matrix of the vectors $w_1, \ldots, w_k \in H$.

Observe that

$$\begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_n \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_n \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \ldots & \langle u_m, v_n \rangle \end{pmatrix}$$

is not a Gram matrix!

A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

Recall that ξ is an n-dimensional Gaussian random vector with respect to the "parameters" μ and C (short: $\xi \sim N_n(\mu, C)$) if and only if for all $a \in \mathbb{R}^n$ there exists $\eta_a \sim N_1(0, 1)$ such that

$$\langle a, \xi \rangle = \sum_{i=1}^{n} a_i \xi_i = \langle a, \mu \rangle + \sqrt{\langle a, Ca \rangle}\,\eta_a = \langle a, \mu \rangle + \|C^{1/2} a\| \eta_a \,.$$

A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

Recall that ξ is an n-dimensional Gaussian random vector with respect to the "parameters" μ and C (short: $\xi \sim N_n(\mu, C)$) if and only if for all $a \in \mathbb{R}^n$ there exists $\eta_a \sim N_1(0, 1)$ such that

$$\langle a, \xi \rangle = \sum_{i=1}^n a_i \xi_i = \langle a, \mu \rangle + \sqrt{\langle a, Ca \rangle}\,\eta_a = \langle a, \mu \rangle + \|C^{1/2} a\| \eta_a \,.$$

Note that we don't require here that C is invertible!

A refresher of a few definitions III

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

Recall that ξ is an n-dimensional Gaussian random vector with respect to the "parameters" μ and C (short: $\xi \sim N_n(\mu, C)$) if and only if for all $a \in \mathbb{R}^n$ there exists $\eta_a \sim N_1(0, 1)$ such that

$$\langle a, \xi \rangle = \sum_{i=1}^{n} a_i \xi_i = \langle a, \mu \rangle + \sqrt{\langle a, Ca \rangle}\,\eta_a = \langle a, \mu \rangle + \|C^{1/2} a\| \eta_a \,.$$

Note that we don't require here that C is invertible! Following Feller, the matrix $\mathbb{V}(\xi)$ defined as

$$\mathbb{V}(\xi)_{ij} := \mathbb{E}[\xi_i \xi_j] - \mathbb{E}[\xi_i]\mathbb{E}[\xi_j] \stackrel{(!)}{=} C_{ij} \qquad (i, j \in [n])$$

is known as the *variance matrix of the Gaussian random vector* ξ.

Characterisation of $PSD(n;\mathbb{R})$ I

Proposition

Let $n \in \mathbb{N}$ and $S \in \mathbb{M}(n \times n;\mathbb{R})$. TFAE:

Characterisation of $PSD(n;\mathbb{R})$ I

Proposition

Let $n \in \mathbb{N}$ *and* $S \in \mathbb{M}(n \times n;\mathbb{R})$. *TFAE:*

(i) S *is the variance matrix of some Gaussian random vector.*

Characterisation of $PSD(n;\mathbb{R})$ I

Proposition

Let $n \in \mathbb{N}$ *and* $S \in \mathbb{M}(n \times n;\mathbb{R})$. *TFAE:*

(i) S *is the variance matrix of some Gaussian random vector.*

(ii) $S = \mathbb{E}[\xi\xi^\top]$ *for some Gaussian random vector* $\xi \sim N_n(0, C)$.

Characterisation of $PSD(n;\mathbb{R})$ I

Proposition

Let $n \in \mathbb{N}$ and $S \in \mathbb{M}(n \times n;\mathbb{R})$. TFAE:

(i) *S is the variance matrix of some Gaussian random vector.*

(ii) *$S = \mathbb{E}[\xi\xi^\top]$ for some Gaussian random vector $\xi \sim N_n(0, C)$.*

(iii) *$S \in PSD(n;\mathbb{R})$.*

Characterisation of $PSD(n;\mathbb{R})$ I

Proposition

Let $n \in \mathbb{N}$ and $S \in \mathbb{M}(n \times n; \mathbb{R})$. TFAE:

(i) *S is the variance matrix of some Gaussian random vector.*

(ii) *$S = \mathbb{E}[\xi\xi^\top]$ for some Gaussian random vector $\xi \sim N_n(0, C)$.*

(iii) *$S \in PSD(n;\mathbb{R})$.*

(iv) *There exists a Hilbert space L and vectors $z_1, \ldots, z_n$ in L such that*

$$S = \Gamma_L(z, z) = \sum_{l=1}^{n} z_l z_l^\top = \sum_{l=1}^{n} \mathit{diag}(z_l)\, J\, \mathit{diag}(z_l)$$

where $z := (z_1, z_2, \ldots, z_n)^\top \in L^n$ and $J_{kl} := 1$ for all $k, l \in [n]$.

Characterisation of $PSD(n;\mathbb{R})$ II

This leads us straightforwardly to the following important block matrix representation of positive semidefinite matrices:

Characterisation of $PSD(n;\mathbb{R})$ II

This leads us straightforwardly to the following important block matrix representation of positive semidefinite matrices:

Corollary

Let $m, n \in \mathbb{N}$ *and* $S \in PSD(m+n;\mathbb{R})$ *Then there is a (finite-dimensional) Hilbert space* H *and vectors* $u_1, u_2, \dots u_m, v_1, v_2, \dots v_n \in H$ *such that*

$$S = \Gamma_H(w,w) = \begin{pmatrix} \Gamma_H(u,u) & \Gamma_H(u,v) \\ \Gamma_H(u,v)^\top & \Gamma_H(v,v) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\xi\xi^\top] & \mathbb{E}[\xi\eta^\top] \\ \mathbb{E}[\xi\eta^\top]^\top & \mathbb{E}[\eta\eta^\top] \end{pmatrix}$$

where $w_i := u_i$ *if* $1 \leq i \leq m$ *and* $w_i := v_{i-m}$ *if* $m+1 \leq i \leq m+n$ *and* $(\xi_1, \dots \xi_m, \eta_1, \dots, \eta_n)^\top \sim N_{m+n}(0, S)$.

Characterisation of $PSD(n;\mathbb{R})$ II

This leads us straightforwardly to the following important block matrix representation of positive semidefinite matrices:

Corollary

Let $m, n \in \mathbb{N}$ *and* $S \in PSD(m+n;\mathbb{R})$ *Then there is a (finite-dimensional) Hilbert space* H *and vectors* $u_1, u_2, \ldots u_m, v_1, v_2, \ldots v_n \in H$ *such that*

$$S = \Gamma_H(w,w) = \begin{pmatrix} \Gamma_H(u,u) & \Gamma_H(u,v) \\ \Gamma_H(u,v)^\top & \Gamma_H(v,v) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[\xi\xi^\top] & \mathbb{E}[\xi\eta^\top] \\ \mathbb{E}[\xi\eta^\top]^\top & \mathbb{E}[\eta\eta^\top] \end{pmatrix}$$

where $w_i := u_i$ *if* $1 \leq i \leq m$ *and* $w_i := v_{i-m}$ *if* $m+1 \leq i \leq m+n$ *and* $(\xi_1, \ldots \xi_m, \eta_1, \ldots, \eta_n)^\top \sim N_{m+n}(0,S)$.

Moreover, we have:

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ *and* $\Sigma = \left(\sigma_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{R})$. *TFAE:*

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ *and* $\Sigma = \left(\sigma_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{R})$. *TFAE:*

(i) $\Sigma \in PSD(n; \mathbb{R})$ *and* $\sigma_{ii} = 1$ *for all* $i \in [n]$.

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ *and* $\Sigma = \left(\sigma_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{R})$. *TFAE:*

(i) $\Sigma \in PSD(n; \mathbb{R})$ *and* $\sigma_{ii} = 1$ *for all* $i \in [n]$.

(ii) $\Sigma \in C(n; \mathbb{R})$.

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ *and* $\Sigma = \left(\sigma_{ij}\right) \in \mathbb{M}(n \times n; \mathbb{R})$. *TFAE:*

(i) $\Sigma \in PSD(n; \mathbb{R})$ *and* $\sigma_{ii} = 1$ *for all* $i \in [n]$.

(ii) $\Sigma \in C(n; \mathbb{R})$.

(iii) $\Sigma = \Gamma_{l_2^n}(x, x) = \sum_{i=1}^n x_i x_i^\top = \sum_{i=1}^n \mathit{diag}(x_i)\, J\, \mathit{diag}(x_i)$ *for some* $x = (x_1, \ldots, x_n)^\top \in \left(S^{n-1}\right)^n$.

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ *and* $\Sigma = \big(\sigma_{ij}\big) \in \mathbb{M}(n \times n; \mathbb{R})$*. TFAE:*

(i) $\Sigma \in PSD(n; \mathbb{R})$ *and* $\sigma_{ii} = 1$ *for all* $i \in [n]$*.*

(ii) $\Sigma \in C(n; \mathbb{R})$*.*

(iii) $\Sigma = \Gamma_{l_2^n}(x, x) = \sum_{i=1}^{n} x_i x_i^\top = \sum_{i=1}^{n}$ *diag*(x_i) *J diag*(x_i) *for some* $x = (x_1, \ldots, x_n)^\top \in \big(S^{n-1}\big)^n$*.*

(iv) Σ *is a correlation matrix, induced by some* n*-dimensional Gaussian random vector.*

Geometry of correlation matrices

Observation

Let $n \in \mathbb{N}$ and $\Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R})$. TFAE:

(i) $\Sigma \in PSD(n; \mathbb{R})$ *and* $\sigma_{ii} = 1$ *for all* $i \in [n]$.

(ii) $\Sigma \in C(n; \mathbb{R})$.

(iii) $\Sigma = \Gamma_{l_2^n}(x, x) = \sum_{i=1}^n x_i x_i^\top = \sum_{i=1}^n \textit{diag}(x_i)\, J\, \textit{diag}(x_i)$ *for some* $x = (x_1, \ldots, x_n)^\top \in \left(S^{n-1}\right)^n$.

(iv) Σ *is a correlation matrix, induced by some* n*-dimensional Gaussian random vector.*

In particular, condition (i) *implies that* $\sigma_{ij} \in [-1, 1]$ *for all* $i, j \in [n]$.

Lurking correlation matrices in GT I

Let $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$.
Consider $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, defined as

Lurking correlation matrices in GT I

Let $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$.
Consider $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, defined as

$$\Gamma_H(u, v)_{ij} := \langle u_i, v_j \rangle_H \qquad \big((i,j) \in [m] \times [n]\big).$$

Lurking correlation matrices in GT I

Let $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$. Consider $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, defined as

$$\Gamma_H(u, v)_{ij} := \langle u_i, v_j \rangle_H \qquad \big((i,j) \in [m] \times [n]\big).$$

Put $w := (u^\top, v^\top)^\top \equiv (u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)^\top$. Then $w \in S_H^{m+n}$.

Lurking correlation matrices in GT I

Let $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$. Consider $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, defined as

$$\Gamma_H(u, v)_{ij} := \langle u_i, v_j \rangle_H \qquad \big((i, j) \in [m] \times [n]\big).$$

Put $w := (u^\top, v^\top)^\top \equiv (u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)^\top$. Then $w \in S_H^{m+n}$.

Consequently we have canonically "embedded" our (column) vector $w \in S_H^{m+n}$ into the following symmetric and positive semidefinite $(m+n) \times (m+n)$-correlation matrix:

Lurking correlation matrices in GT I

Let $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$.
Consider $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, defined as

$$\Gamma_H(u, v)_{ij} := \langle u_i, v_j \rangle_H \qquad \big((i, j) \in [m] \times [n]\big).$$

Put $w := (u^\top, v^\top)^\top \equiv (u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)^\top$. Then $w \in S_H^{m+n}$.
Consequently we have canonically "embedded" our (column) vector $w \in S_H^{m+n}$ into the following symmetric and positive semidefinite $(m + n) \times (m + n)$-correlation matrix:

$$\Gamma_H(w, w) = \begin{pmatrix} \Gamma_H(u, u) & \Gamma_H(u, v) \\ \Gamma_H(u, v)^\top & \Gamma_H(v, v) \end{pmatrix}$$

Lurking correlation matrices in GT II

Moreover,

$$\Gamma_{\mathbb{R}}(p,q) = pq^{\top} \text{ for all } p \in \mathbb{R}^m, q \in \mathbb{R}^n$$

and

Lurking correlation matrices in GT II

Moreover,

$$\Gamma_{\mathbb{R}}(p,q) = pq^\top \text{ for all } p \in \mathbb{R}^m, q \in \mathbb{R}^n$$

and

$$\Gamma_{\mathbb{R}}(x,x) = xx^\top = \begin{pmatrix} pp^\top & pq^\top \\ qp^\top & qq^\top \end{pmatrix} \text{ for all } x := (p^\top, q^\top)^\top \in \mathbb{R}^{m+n}.$$

Geometry of correlation matrices II

Exercise

Let $k \in \mathbb{N}$. Then the sets $\left\{S : S = xx^\top \text{ for some } x \in \{-1,1\}^k\right\}$ and $\left\{\Sigma : \Sigma \in C(k;\mathbb{R}) \text{ and } rk(\Sigma) = 1\right\}$ coincide.

Geometry of correlation matrices III

Proposition (K. R. Parthasarathy (2002))

Let $k \in \mathbb{N}$. $C(k; \mathbb{R})$ is a compact and convex subset of the k^2-dimensional vector space $\mathbb{M}(k \times k; \mathbb{R})$. Any $k \times k$-correlation matrix of rank 1 is an extremal point of the set $C(k; \mathbb{R})$.

Geometry of correlation matrices III

Proposition (K. R. Parthasarathy (2002))

Let $k \in \mathbb{N}$. $C(k;\mathbb{R})$ is a compact and convex subset of the k^2-dimensional vector space $\mathbb{M}(k \times k;\mathbb{R})$. Any $k \times k$-correlation matrix of rank 1 is an extremal point of the set $C(k;\mathbb{R})$.

In particular, the (finite) set of all $k \times k$-correlation matrices of rank 1 is not convex.

Geometry of correlation matrices III

Proposition (K. R. Parthasarathy (2002))

Let $k \in \mathbb{N}$. $C(k;\mathbb{R})$ is a compact and convex subset of the k^2-dimensional vector space $\mathbb{M}(k \times k;\mathbb{R})$. Any $k \times k$-correlation matrix of rank 1 is an extremal point of the set $C(k;\mathbb{R})$.

In particular, the (finite) set of all $k \times k$-correlation matrices of rank 1 is not convex.

Let $k \in \mathbb{N}$. Put

$$C_1(k;\mathbb{R}) := \left\{\Sigma : \Sigma \in C(k;\mathbb{R}) \text{ and } \mathrm{rk}(\Sigma) = 1\right\}.$$

Canonical block injection of A I

A naturally appearing question is the following:

Canonical block injection of A I

A naturally appearing question is the following:

Having gained - important - additional structure by “enlarging” the $m \times n$-matrix $\Gamma_H(u, v)$ to a $(m+n) \times (m+n)$-correlation matrix, how could this gained information be used to rewrite Grothendieck’s inequality accordingly?

Canonical block injection of A I

A naturally appearing question is the following:

Having gained - important - additional structure by “enlarging” the $m \times n$-matrix $\Gamma_H(u, v)$ to a $(m + n) \times (m + n)$-correlation matrix, how could this gained information be used to rewrite Grothendieck’s inequality accordingly? To answer this question, let us also “embed” the $m \times n$-matrix A suitably!

Canonical block injection of A I

A naturally appearing question is the following:

Having gained - important - additional structure by "enlarging" the $m \times n$-matrix $\Gamma_H(u, v)$ to a $(m+n) \times (m+n)$-correlation matrix, how could this gained information be used to rewrite Grothendieck's inequality accordingly? To answer this question, let us also "embed" the $m \times n$-matrix A suitably!

Definition

Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{R})$ arbitrary. Put

$$\widehat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

Let us call $\mathbb{M}(m \times n; \mathbb{R}) \ni \widehat{A}$ the canonical block injection of A.

Canonical block injection of A I

A naturally appearing question is the following:

Having gained - important - additional structure by "enlarging" the $m \times n$-matrix $\Gamma_H(u, v)$ to a $(m+n) \times (m+n)$-correlation matrix, how could this gained information be used to rewrite Grothendieck's inequality accordingly? To answer this question, let us also "embed" the $m \times n$-matrix A suitably!

Definition

Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{R})$ arbitrary. Put

$$\widehat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

Let us call $\mathbb{M}(m \times n; \mathbb{R}) \ni \widehat{A}$ the canonical block injection of A.

Observe that $\widehat{A}$ is symmetric, implying that $\widehat{A} = \widehat{A}^\top$.

A further equivalent rewriting of GT I

Proposition

Let $m, n \in \mathbb{N}$ *and* $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$*. Let* $K > 0$*. TFAE:*

A further equivalent rewriting of GT I

Proposition

Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

(i)

$$\sup_{(u,v) \in S_H^m \times S_H^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max_{(p,q) \in \{-1,1\}^m \times \{-1,1\}^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right|$$

for all Hilbert spaces H over $\mathbb{R}$.

A further equivalent rewriting of GT I

Proposition

Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

(i)

$$\sup_{(u,v)\in S_H^m \times S_H^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max_{(p,q)\in\{-1,1\}^m \times \{-1,1\}^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right|$$

for all Hilbert spaces H over $\mathbb{R}$.

(ii)

$$\sup_{\Gamma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Gamma \rangle| \leq K \max_{\substack{\Sigma \in C(m+n;\mathbb{R}) \\ rk(\Sigma)=1}} |\langle \widehat{A}, \Sigma \rangle| \, .$$

A further equivalent rewriting of GT II

Proposition

Let $m, n \in \mathbb{N}$ and $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. Let $K > 0$. TFAE:

(i)

$$\max_{(u,v) \in S_H^m \times S_H^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max_{(p,q) \in \{-1,1\}^m \times \{-1,1\}^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right|$$

for all Hilbert spaces H over $\mathbb{R}$.

(ii)

$$\max_{\Gamma \in C(m+n; \mathbb{R})} |\langle \widehat{A}, \Gamma \rangle| \leq K \max_{\Sigma \in C_1(m+n; \mathbb{R})} |\langle \widehat{A}, \Sigma \rangle| \,.$$

A further equivalent rewriting of GT II

Proposition

Let $m, n \in \mathbb{N}$ *and* $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. *Let* $K > 0$. *TFAE:*

(i)

$$\max_{(u,v)\in S_H^m \times S_H^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \right| \leq K \max_{(p,q)\in\{-1,1\}^m \times \{-1,1\}^n} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \right|$$

for all Hilbert spaces H *over* $\mathbb{R}$.

(ii)

$$\max_{\Gamma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Gamma \rangle| \leq K \max_{\Sigma \in C_1(m+n;\mathbb{R})} |\langle \widehat{A}, \Sigma \rangle| .$$

We don't know whether condition (ii) holds for all matrices in $\mathbb{M}((m+n) \times (m+n); \mathbb{R})$.

GT versus NP-hard optimisation

Observation

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

$$\max_{\Gamma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Gamma \rangle|$$

GT versus NP-hard optimisation

Observation

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

$$\max_{\Gamma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Gamma \rangle|$$

On the right side: an NP-hard, non-convex combinatorial (Boolean) optimisation problem:

$$\max_{\substack{\Sigma \in C(m+n;\mathbb{R}) \\ rk(\Sigma)=1}} |\langle \widehat{A}, \Sigma \rangle|$$

GT versus NP-hard optimisation

Observation

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

$$\max_{\Gamma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Gamma \rangle|$$

On the right side: an NP-hard, non-convex combinatorial (Boolean) optimisation problem:

$$\max_{\substack{\Sigma \in C(m+n;\mathbb{R}) \\ rk(\Sigma)=1}} |\langle \widehat{A}, \Sigma \rangle|$$

Thus, Grothendieck's constant $K_G^{\mathbb{R}}$ is precisely the "integrality gap"; i. e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!

1. A very short glimpse at A. Grothendieck's work in functional analysis
2. Grothendieck's inequality in matrix formulation
3. Grothendieck's inequality rewritten
4. Grothendieck's inequality and correlation matrices
5. Towards a calculation of Grothendieck's constant $K_G^{\mathbb{R}}$
6. Grothendieck's inequality and its relation to non-locality in quantum mechanics

Schur product and the matrix $f[A]$

Definition

Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i,j) \in [m] \times [n]$.

Schur product and the matrix $f[A]$

Definition

Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i,j) \in [m] \times [n]$. Define $f[A] \in \mathbb{M}(m \times n; \mathbb{R})$ - entrywise - as $f[A]_{ij} := f(a_{ij})$ for all $(i,j) \in [m] \times [n]$.

Schur product and the matrix $f[A]$

Definition

Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i,j) \in [m] \times [n]$. Define $f[A] \in \mathbb{M}(m \times n; \mathbb{R})$ - entrywise - as $f[A]_{ij} := f(a_{ij})$ for all $(i,j) \in [m] \times [n]$.

Guiding Example

The Schur product (or Hadamard product)

$$(a_{ij}) * (b_{ij}) := (a_{ij} b_{ij})$$

of matrices (a_{ij}) and (b_{ij}) leads to $f[A]$, where $f(x) := x^2$.

Schur product and the matrix $f[A]$

Definition

Let $\emptyset \neq I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in I$ for all $(i,j) \in [m] \times [n]$. Define $f[A] \in \mathbb{M}(m \times n; \mathbb{R})$ - entrywise - as $f[A]_{ij} := f(a_{ij})$ for all $(i,j) \in [m] \times [n]$.

Guiding Example

The Schur product (or Hadamard product)

$$(a_{ij}) * (b_{ij}) := (a_{ij} b_{ij})$$

of matrices (a_{ij}) and (b_{ij}) leads to $f[A]$, where $f(x) := x^2$.

Remark

The notation "$f[A]$" is used to highlight the difference between the matrix $f(A)$ originating from the spectral representation of A (for normal matrices A) and the matrix $f[A]$, defined as above !

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$.

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$*, where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign : $\mathbb{R} \longrightarrow \{-1, 1\}$*, defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$*. Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$*,* $corr(\xi, \eta) = E[\xi\eta] = \rho$*, and*

$$\mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)]$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign : $\mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= 1\!\!1_{[0,\infty)} - 1\!\!1_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

$$\mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)] \quad =$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign : $\mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

$$\mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)] \quad = \quad 4\mathbb{P}(\xi \geq 0, \eta \geq 0) - 1$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign : $\mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

$$\begin{aligned} \mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)] &= 4\mathbb{P}(\xi \geq 0, \eta \geq 0) - 1 \\ &= \end{aligned}$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$*, where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$*, defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$*. Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$*,* $corr(\xi, \eta) = E[\xi\eta] = \rho$*, and*

$$\begin{aligned} \mathbb{E}[\textit{sign}(\xi)\textit{sign}(\eta)] &= 4\mathbb{P}(\xi \geq 0, \eta \geq 0) - 1 \\ &= \frac{2}{\pi} \arcsin\left(\mathbb{E}[\xi\eta]\right) \end{aligned}$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

$$\begin{aligned} \mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)] &= 4\mathbb{P}(\xi \geq 0, \eta \geq 0) - 1 \\ &= \frac{2}{\pi} \arcsin\left(\mathbb{E}[\xi\eta]\right) = \end{aligned}$$

Grothendieck's identity I

How can we link an NP-hard non-convex Boolean optimisation problem and its convex SDP relaxation?

Theorem (Grothendieck's identity - T. S. Stieltjes (1889))

Let $-1 \leq \rho \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Theta_\rho)$, *where*

$$\Theta_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign* $:= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$. *Then* $\xi \sim N_1(0,1), \eta \sim N_1(0,1)$, $corr(\xi, \eta) = E[\xi\eta] = \rho$, *and*

$$\begin{aligned} \mathbb{E}[\mathit{sign}(\xi)\mathit{sign}(\eta)] &= 4\mathbb{P}(\xi \geq 0, \eta \geq 0) - 1 \\ &= \frac{2}{\pi} \arcsin\left(\mathbb{E}[\xi\eta]\right) = \frac{2}{\pi} \arcsin(\rho). \end{aligned}$$

Grothendieck's identity II

Corollary

Let $k \in \mathbb{N}$. Let $\Theta \in C(k; \mathbb{R})$ an arbitrarily given correlation matrix. Then there exists a Gaussian random vector $\xi \sim N_k(0, \Theta)$ such that

$$\frac{2}{\pi} \arcsin[\Theta] = \mathbb{E}\left[\Sigma(\xi)\right] ,$$

where

$$\Sigma(\xi(\omega))_{ij} := \textit{sign}(\xi_i(\omega))\textit{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$, and for all $i, j \in [k]$.

Grothendieck's identity II

Corollary

Let $k \in \mathbb{N}$. Let $\Theta \in C(k;\mathbb{R})$ an arbitrarily given correlation matrix. Then there exists a Gaussian random vector $\xi \sim N_k(0,\Theta)$ such that

$$\frac{2}{\pi}\arcsin[\Theta] = \mathbb{E}\big[\Sigma(\xi)\big]\,,$$

where

$$\Sigma(\xi(\omega))_{ij} := \textit{sign}(\xi_i(\omega))\textit{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$, and for all $i,j \in [k]$. $\Sigma(\xi(\omega))$ is a correlation matrix of rank 1 for all $\omega \in \Omega$, and we have

Grothendieck's identity II

Corollary

Let $k \in \mathbb{N}$. *Let* $\Theta \in C(k;\mathbb{R})$ *an arbitrarily given correlation matrix. Then there exists a Gaussian random vector* $\xi \sim N_k(0,\Theta)$ *such that*

$$\frac{2}{\pi}\arcsin[\Theta] = \mathbb{E}\left[\Sigma(\xi)\right],$$

where

$$\Sigma(\xi(\omega))_{ij} := \mathit{sign}(\xi_i(\omega))\mathit{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$, *and for all* $i,j \in [k]$. $\Sigma(\xi(\omega))$ *is a correlation matrix of rank* 1 *for all* $\omega \in \Omega$, *and we have*

$$\begin{aligned}\max_{\substack{\Sigma \in C(k;\mathbb{R})\\ rank(\Sigma)=1}} |\langle \widehat{A}, \Sigma\rangle| &\geq \mathbb{E}\left[|\langle \widehat{A}, \Sigma(\xi)\rangle|\right]\\ &\geq |\langle \widehat{A}, \mathbb{E}\left[\Sigma(\xi)\right]\rangle| = \frac{2}{\pi}|\langle \widehat{A}, \arcsin[\Theta]\rangle|\,.\end{aligned}$$

Grothendieck's identity III

Corollary

Let $m, n \in \mathbb{N}$ and H be an arbitrary Hilbert space. Let $u \in S_H^m$ and $v \in S_H^n$. Then

$$\Theta_{u,v} := \frac{2}{\pi} \arcsin[\Gamma_H(u,v)] = \mathbb{E}\big[\mathit{sign}(\xi)\,\mathit{sign}(\eta)^\top\big]$$

for some Gaussian random vectors $\xi \sim N_m\big(0, \Theta_{u,u}\big)$ and $\eta \sim N_n\big(0, \Theta_{v,v}\big)$.

Grothendieck's identity III

Corollary

Let $m, n \in \mathbb{N}$ and H be an arbitrary Hilbert space. Let $u \in S_H^m$ and $v \in S_H^n$. Then

$$\Theta_{u,v} := \frac{2}{\pi} \arcsin[\Gamma_H(u,v)] = \mathbb{E}\big[\mathit{sign}(\xi)\,\mathit{sign}(\eta)^\top\big]$$

for some Gaussian random vectors $\xi \sim N_m\big(0, \Theta_{u,u}\big)$ and $\eta \sim N_n\big(0, \Theta_{v,v}\big)$.

More generally, we have

Grothendieck's identity III

Corollary

Let $m, n \in \mathbb{N}$ and H be an arbitrary Hilbert space. Let $u \in S_H^m$ and $v \in S_H^n$. Then

$$\Theta_{u,v} := \frac{2}{\pi} \arcsin[\Gamma_H(u,v)] = \mathbb{E}\left[\textit{sign}(\xi)\,\textit{sign}(\eta)^\top\right]$$

for some Gaussian random vectors $\xi \sim N_m\big(0, \Theta_{u,u}\big)$ and $\eta \sim N_n\big(0, \Theta_{v,v}\big)$.

More generally, we have

Proposition (Schoenberg (1942))

Let $k \in \mathbb{N}$ and $0 < r \leq \infty$. Let Θ be an arbitrary $(k \times k)$-correlation matrix. Let f be a function that admits a power series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for some sequence $(a_n) \subseteq [0, \infty)$ on $(-r, r)$. Let $0 < f(1)$ and put $f^{[1]} := \frac{1}{f(1)} f$. Then $f^{[1]}\big[\Theta\big]$ again is a $(k \times k)$-correlation matrix.

Grothendieck's identity IV

Since for all $\rho \in [-1, 1]$

$$\arcsin(\rho) = \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\rho^{2n+1}}{(2n+1)!} = \rho + \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\rho^{2n+1}}{(2n+1)!}$$

it follows that (in general)

Grothendieck's identity IV

Since for all $\rho \in [-1, 1]$

$$\arcsin(\rho) = \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\rho^{2n+1}}{(2n+1)!} = \rho + \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{\rho^{2n+1}}{(2n+1)!}$$

it follows that (in general)

$$\begin{aligned} \max_{\substack{\Sigma \in C(k;\mathbb{R}) \\ rank(\Sigma)=1}} |\langle \widehat{A}, \Sigma \rangle| &\geq \frac{2}{\pi} |\langle \widehat{A}, \arcsin[\Theta] \rangle| \\ &= \frac{2}{\pi} \left| \langle \widehat{A}, \Theta \rangle + \sum_{n=1}^{\infty} \frac{1}{4^n (2n+1)!} \binom{2n}{n} \langle \widehat{A}, [\Theta]^{2n+1} \rangle \right| . \end{aligned}$$

Exercise

Let $k \in \mathbb{N}$ *and* $M \in PSD(k;\mathbb{R})$.

Exercise

Let $k \in \mathbb{N}$ *and* $M \in PSD(k;\mathbb{R})$.

Show that

$$\arcsin[M] - M \in PSD(k;\mathbb{R})\,.$$

Exercise

Let $k \in \mathbb{N}$ *and* $M \in PSD(k;\mathbb{R})$.

Show that

$$\arcsin[M] - M \in PSD(k;\mathbb{R})\,.$$

How can we treat the difficult handling of the remaining part

$$\sum_{n=1}^{\infty} \frac{1}{4^n(2n+1)!}\binom{2n}{n}\langle \widehat{A}, [\Theta]^{2n+1}\rangle$$

(which unfortunately “sits inside” an absolute value)?

Grothendieck's identity V

A seemingly fruitful and different approach is the following:

Grothendieck's identity V

A seemingly fruitful and different approach is the following:

(i) Transform an arbitrarily given correlation matrix Θ_0 non-linearly - and entrywise - to another correlation matrix $\Theta_1 := \Phi[\Theta_0]$ for some $\Phi : C(k;\mathbb{R}) \longrightarrow C(k;\mathbb{R})$ such that this non-linear transformation Φ strongly reduces the impact of the $\arcsin$ function (up to a given small error).

Grothendieck's identity V

A seemingly fruitful and different approach is the following:

(i) Transform an arbitrarily given correlation matrix Θ_0 non-linearly - and entrywise - to another correlation matrix $\Theta_1 := \Phi[\Theta_0]$ for some $\Phi : C(k;\mathbb{R}) \longrightarrow C(k;\mathbb{R})$ such that this non-linear transformation Φ strongly reduces the impact of the $\arcsin$ function (up to a given small error).

(ii) Apply Grothendieck's identity to the so obtained correlation matrix Θ_1 and apply the estimation above - to $\arcsin[\Theta_1]$.

Grothendieck's identity V

A seemingly fruitful and different approach is the following:

(i) Transform an arbitrarily given correlation matrix Θ_0 non-linearly - and entrywise - to another correlation matrix $\Theta_1 := \Phi[\Theta_0]$ for some $\Phi : C(k;\mathbb{R}) \longrightarrow C(k;\mathbb{R})$ such that this non-linear transformation Φ strongly reduces the impact of the $\arcsin$ function (up to a given small error).

(ii) Apply Grothendieck's identity to the so obtained correlation matrix Θ_1 and apply the estimation above - to $\arcsin[\Theta_1]$.

(iii) A reiteration of the steps (i) and (ii) could lead to an iterative algorithm which might converge to a "suitable" - upper - bound of $K_G^{\mathbb{R}}$.

1. A very short glimpse at A. Grothendieck's work in functional analysis
2. Grothendieck's inequality in matrix formulation
3. Grothendieck's inequality rewritten
4. Grothendieck's inequality and correlation matrices
5. Towards a calculation of Grothendieck's constant $K_G^{\mathbb{R}}$
6. Grothendieck's inequality and its relation to non-locality in quantum mechanics

Modelling quantum correlation I

Following Tsirelson's thoughts we consider two sets, the set of all "classical" (local) $(m \times n)$-cross-correlation matrices and the set of all $(m \times n)$-quantum correlation matrices:

Modelling quantum correlation I

Following Tsirelson's thoughts we consider two sets, the set of all "classical" (local) $(m \times n)$-cross-correlation matrices and the set of all $(m \times n)$-quantum correlation matrices:

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ("classical" Kolmogorovian) probability space. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$. $A \in C_{\text{loc}}(m \times n; \mathbb{R})$ iff $a_{ij} = \mathbb{E}_{\mathbb{P}}[X_i Y_j]$, where $X_i, Y_j : \Omega \longrightarrow [-1, 1]$ are random variables - all defined on the same given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Modelling quantum correlation I

Following Tsirelson's thoughts we consider two sets, the set of all "classical" (local) $(m \times n)$-cross-correlation matrices and the set of all $(m \times n)$-quantum correlation matrices:

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ("classical" Kolmogorovian) probability space. Let $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{R})$. $A \in C_{\text{loc}}(m \times n; \mathbb{R})$ iff $a_{ij} = \mathbb{E}_{\mathbb{P}}\big[X_i Y_j\big]$, where $X_i, Y_j : \Omega \longrightarrow [-1, 1]$ are random variables - all defined on the same given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) Let $A = \big(a_{ij}\big) \in \mathbb{M}(m \times n; \mathbb{R})$. $A \in QC(m \times n; \mathbb{R})$ iff there are $k, l \in \mathbb{N}$, a density matrix ρ on $\mathcal{B}\big(H_{k,l}\big)$, where $H_{k,l} := \mathbb{C}^k \otimes \mathbb{C}^l$, and linear operators $A_i \in \mathcal{B}\big(\mathbb{C}^k\big)$, $B_j \in \mathcal{B}\big(\mathbb{C}^l\big)$ such that $\|A_i\| \leq 1$, $\|B_j\| \leq 1$ and

$$a_{ij} = \langle \rho, A_i \otimes B_j \rangle = \mathrm{tr}\big(\rho(A_i \otimes B_j)\big) = \mathrm{tr}\big(\rho(A_i \otimes E_l)(E_k \otimes B_j)\big)$$

for all $(i, j) \in [m] \times [n]$.

Modelling quantum correlation II

Does $QC(m \times n; \mathbb{R})$ relate to our previous investigation of the left side of GT?

Modelling quantum correlation II

Does $QC(m \times n; \mathbb{R})$ relate to our previous investigation of the left side of GT?

In fact! Tsirelson unrevealed the following characterisation:

Modelling quantum correlation II

Does $QC(m \times n; \mathbb{R})$ relate to our previous investigation of the left side of GT?

In fact! Tsirelson unrevealed the following characterisation:

Theorem (Tsirelson (1987, 1993))

Let $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. *TFAE:*

(i) $A \in QC(m \times n; \mathbb{R})$.

(ii) $A = \Gamma_{l_2^k}(u, v)$ *for some* $k \in \mathbb{N}$ *and some* $u \in \left(S^{k-1}\right)^m$ *and* $v \in \left(S^{k-1}\right)^n$.

Modelling quantum correlation III

$$\Gamma_{l_2^k}(u,v) = \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}$$

Modelling quantum correlation III

$$\Gamma_{l_2^k}(u,v) = \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}$$

$\Gamma_{l_2^k}(u,v) = UV$ is the product of the matrices $U : l_2^k \longrightarrow l_\infty^m$ and $V : l_1^n \longrightarrow l_2^k$, where

$$V := \left(v_1 \,\vdots\, v_2 \,\vdots\, \dots \,\vdots\, v_n\right) \text{ and } U := \begin{pmatrix} u_1^\top \\ u_2^\top \\ \vdots \\ u_m^\top \end{pmatrix}.$$

Modelling quantum correlation IV

Hence, we see that if $u \in S_H^m$ and $v \in S_H^n$ one can canonically associate a linear operator to the $(m \times n)$-matrix $\Gamma_H(u,v)$ which factors through the Hilbert space $H := l_2^k$ such that $\Gamma_H(u,v) = UV$ for some $(m \times k)$-matrix U and some $(k \times n)$-matrix V, satisfying

$$\gamma_2\big(\Gamma_H(u,v)\big) \leq \|U\|_{2,\infty} \cdot \|V\|_{1,2} \leq 1 \ :$$

$$\begin{tikzcd} l_1^n \arrow[rr, "\Gamma_H(u,v)"] \arrow[dr, "V"'] & & l_\infty^m \\ & H \arrow[ur, "U"'] & \end{tikzcd}$$

Modelling quantum correlation V

Theorem (Grothendieck (1953), Pisier (2001), Tsirelson (1987))

Let H be a separable Hilbert space and $m, n \in \mathbb{N}$. Let $u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$. Then

$$\begin{aligned} \Gamma_H(u, v) &\in K_G^{\mathbb{R}} \, \mathit{cx}\Big(\big\{pq^\top : p \in \{-1,1\}^m, q \in \{-1,1\}^n\big\}\Big) \\ &= K_G^{\mathbb{R}} \, C_{\mathit{loc}}(m \times n; \mathbb{R}) \, . \end{aligned}$$

Modelling quantum correlation V

Theorem (Grothendieck (1953), Pisier (2001), Tsirelson (1987))

Let H be a separable Hilbert space and $m, n \in \mathbb{N}$. Let $u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$. Then

$$\begin{aligned} \Gamma_H(u, v) &\in K_G^{\mathbb{R}} \, \mathit{cx}\big(\{pq^\top : p \in \{-1, 1\}^m, q \in \{-1, 1\}^n\}\big) \\ &= K_G^{\mathbb{R}} \, C_{\mathit{loc}}(m \times n; \mathbb{R}) \, . \end{aligned}$$

Corollary (Tsirelson (1987, 1993))

Let $m, n \in \mathbb{N}$. Then

$$QC(m \times n; \mathbb{R}) \subseteq K_G^{\mathbb{R}} \, C_{\mathit{loc}}(m \times n; \mathbb{R}) \, .$$

Moreover, $C_{\mathit{loc}}(m \times n; \mathbb{R}) \subseteq QC(m \times n; \mathbb{R})$. The latter set inclusion is strict.

Bell's inequalities and GT I

It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as *Bell's inequalities*.

Bell's inequalities and GT I

It is well-known that it is also experimentally verified that entangled composite quantum systems violate certain relations between correlations - known as *Bell's inequalities*.

Purely in terms of of a very elementary application of classical Kolmogorovian probability theory - and completely independent of any modelling assumptions in physics - Bell's inequalities can be represented in form of an inequality originating from *J. F. Clauser, M. A. Horne, A. Shimony* and *R. A. Holt* in 1969.

Bell's inequalities and GT II

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. Let X_1, X_2, X_3 and X_4 be arbitrary random variables with values in $[-1, 1]$ $\mathbb{P}$-a.s., all defined on Ω. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] - \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2X_3]$$

Bell's inequalities and GT II

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. Let X_1, X_2, X_3 and X_4 be arbitrary random variables with values in $[-1, 1]$ $\mathbb{P}$-a.s., all defined on Ω. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] - \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2X_3]$$

and

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] + \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 + \mathbb{E}_{\mathbb{P}}[X_2X_3] \, .$$

Bell's inequalities and GT II

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. Let X_1, X_2, X_3 and X_4 be arbitrary random variables with values in $[-1, 1]$ $\mathbb{P}$-a.s., all defined on Ω. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1 X_2] - \mathbb{E}_{\mathbb{P}}[X_1 X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2 X_3]$$

and

$$|\mathbb{E}_{\mathbb{P}}[X_1 X_2] + \mathbb{E}_{\mathbb{P}}[X_1 X_3]| \leq 1 + \mathbb{E}_{\mathbb{P}}[X_2 X_3] \,.$$

In particular,

$$|\mathbb{E}_{\mathbb{P}}[X_1 X_2] + \mathbb{E}_{\mathbb{P}}[X_1 X_3] + \mathbb{E}_{\mathbb{P}}[X_4 X_2] - \mathbb{E}_{\mathbb{P}}[X_4 X_3]| \leq 2 \,.$$

Bell's inequalities and GT II

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. Let X_1, X_2, X_3 and X_4 be arbitrary random variables with values in $[-1, 1]$ $\mathbb{P}$-a.s., all defined on Ω. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] - \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2X_3]$$

and

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] + \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 + \mathbb{E}_{\mathbb{P}}[X_2X_3]\,.$$

In particular,

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] + \mathbb{E}_{\mathbb{P}}[X_1X_3] + \mathbb{E}_{\mathbb{P}}[X_4X_2] - \mathbb{E}_{\mathbb{P}}[X_4X_3]| \leq 2\,.$$

Notice that this result holds independently of the choice of the joint distribution of the rv's X_1, X_2, X_3, X_4.

Bell's inequalities and GT II

How is the BCHSH inequality linked with GT?

Bell's inequalities and GT II

How is the BCHSH inequality linked with GT?

To recognise this link let us relabel the random variables first:

Bell's inequalities and GT II

How is the BCHSH inequality linked with GT?

To recognise this link let us relabel the random variables first:

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. Let X_1, X_2, Y_1 and Y_2 be arbitrary random variables with values in $[-1, 1]$ $\mathbb{P}$-a.s., all defined on Ω. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] - \mathbb{E}_{\mathbb{P}}[X_1 Y_2]| \leq 1 - \mathbb{E}_{\mathbb{P}}[Y_1 Y_2]$$

and

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] + \mathbb{E}_{\mathbb{P}}[X_1 Y_2]| \leq 1 + \mathbb{E}_{\mathbb{P}}[Y_1 Y_2] \,.$$

In particular,

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] + \mathbb{E}_{\mathbb{P}}[X_1 Y_2] + \mathbb{E}_{\mathbb{P}}[X_2 Y_1] - \mathbb{E}_{\mathbb{P}}[X_2 Y_2]| \leq 2 \,.$$

Bell's inequalities and GT II

How is the BCHSH inequality linked with GT?

To recognise this link let us relabel the random variables first:

Lemma (BCHSH Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space. Let* X_1, X_2, Y_1 *and* Y_2 *be arbitrary random variables with values in* $[-1, 1]$ $\mathbb{P}$*-a.s., all defined on* Ω*. Then*

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] - \mathbb{E}_{\mathbb{P}}[X_1 Y_2]| \leq 1 - \mathbb{E}_{\mathbb{P}}[Y_1 Y_2]$$

and

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] + \mathbb{E}_{\mathbb{P}}[X_1 Y_2]| \leq 1 + \mathbb{E}_{\mathbb{P}}[Y_1 Y_2] \,.$$

In particular,

$$|\mathbb{E}_{\mathbb{P}}[X_1 Y_1] + \mathbb{E}_{\mathbb{P}}[X_1 Y_2] + \mathbb{E}_{\mathbb{P}}[X_2 Y_1] - \mathbb{E}_{\mathbb{P}}[X_2 Y_2]| \leq 2 \,.$$

In other words:

Observation (BCHSH Inequality in matrix form)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space (in the sense of Kolmogorov).*

Bell's inequalities and GT III

Observation (BCHSH Inequality in matrix form)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space (in the sense of Kolmogorov).*

Put

$$A^{\mathit{Had}} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{(Hadamard matrix} \rightsquigarrow \text{“quantum gate”)}$$

Bell's inequalities and GT III

Observation (BCHSH Inequality in matrix form)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be an arbitrary probability space (in the sense of Kolmogorov).*

Put

$$A^{\mathit{Had}} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \textit{(Hadamard matrix} \rightsquigarrow \textit{"quantum gate")}$$

Then

$$|\langle A^{\mathit{Had}}, \Gamma \rangle| = |\mathit{tr}\big(A^{\mathit{Had}}\Gamma\big)| \leq 2 \;\; \textit{for all} \;\; \Gamma \in C_{\mathit{loc}}(2 \times 2; \mathbb{R})\,.$$

Bell's inequalities and GT IV

Let us turn to the left "quantum correlation side" of GT!

Bell’s inequalities and GT IV

Let us turn to the left “quantum correlation side” of GT!

Theorem (Tsirelson (1980))

Let H be an arbitrary Hilbert space, $u \in S_H^2$ and $v \in S_H^2$. Then

$$|\langle A^{Had}, \Gamma_H(u,v)\rangle| = |tr\big(A^{Had}\,\Gamma_H(u,v)\big)| \le 2\sqrt{2}$$

Bell's inequalities and GT IV

Let us turn to the left "quantum correlation side" of GT!

Theorem (Tsirelson (1980))

Let H be an arbitrary Hilbert space, $u \in S_H^2$ and $v \in S_H^2$. Then

$$|\langle A^{Had}, \Gamma_H(u,v)\rangle| = |tr\big(A^{Had}\,\Gamma_H(u,v)\big)| \leq 2\sqrt{2}$$

Even more holds!

To this end, we recall the main ideas underlying the EPR/Bell-CHSH experiment.

Bell's inequalities and GT V

A source emits in opposite directions two spin $\frac{1}{2}$ particles created from one particle of spin 0. By rotating magnets perpendicular to the directions of the two spin $\frac{1}{2}$ particles, both, Alice and Bob measure the spin in 2 different directions, leading to angles $-\frac{\pi}{2} \leq \alpha_1, \alpha_2 < \frac{\pi}{2}$ for Alice and $-\frac{\pi}{2} \leq \beta_1, \beta_2 < \frac{\pi}{2}$ for Bob. Only one angle per measurement can be chosen on both sides. The outcome of this experiment is a "random" pair of observables belonging to the set

$$\{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2)\} .$$

Any of these observables takes its values in $\{-1, +1\}$.

Bell's inequalities and GT V

A source emits in opposite directions two spin $\frac{1}{2}$ particles created from one particle of spin 0. By rotating magnets perpendicular to the directions of the two spin $\frac{1}{2}$ particles, both, Alice and Bob measure the spin in 2 different directions, leading to angles $-\frac{\pi}{2} \leq \alpha_1, \alpha_2 < \frac{\pi}{2}$ for Alice and $-\frac{\pi}{2} \leq \beta_1, \beta_2 < \frac{\pi}{2}$ for Bob. Only one angle per measurement can be chosen on both sides. The outcome of this experiment is a "random" pair of observables belonging to the set

$$\{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2)\}.$$

Any of these observables takes its values in $\{-1, +1\}$.

Describing this experiment purely in terms of mathematics we immediately recognise that the Bell-Tsirelson constant $2\sqrt{2}$ is attained by the Hadamard matrix, since:

Bell's inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let*
$H \ni x := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$ *("entangled Bell state").*

Bell's inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let* $H \ni x := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$ *("entangled Bell state"). Let*

$$\alpha_1 := \frac{\pi}{2}, \alpha_2 := 0, \beta_1 := -\frac{3\pi}{4} \text{ and } \beta_2 := -\frac{\pi}{4}.$$

Bell's inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let* $H \ni x := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$ *("entangled Bell state"). Let*

$$\alpha_1 := \frac{\pi}{2}, \alpha_2 := 0, \beta_1 := -\frac{3\pi}{4} \text{ and } \beta_2 := -\frac{\pi}{4}\,.$$

Put

$$\Gamma^{EPR} := \begin{pmatrix} \langle x, (A_1 \otimes B_1)x\rangle_H & \langle x, (A_1 \otimes B_2)x\rangle_H \\ \langle x, (A_2 \otimes B_1)x\rangle_H & \langle x, (A_2 \otimes B_2)x\rangle_H \end{pmatrix},$$

where $A_i := R(\alpha_i)$*,* $B_j := R(\beta_j)$ *and*

$$\mathbb{M}(2 \times 2; \mathbb{C}) \ni R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix},$$

Bell's inequalities and GT VI

Theorem (EPR/Bell-CHSH violates Bell and attains $2\sqrt{2}$)

Consider the Hilbert space $H := \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let* $H \ni x := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big)$ *("entangled Bell state"). Let*

$$\alpha_1 := \frac{\pi}{2}, \alpha_2 := 0, \beta_1 := -\frac{3\pi}{4} \text{ and } \beta_2 := -\frac{\pi}{4}.$$

Put

$$\Gamma^{EPR} := \begin{pmatrix} \langle x, (A_1 \otimes B_1)x\rangle_H & \langle x, (A_1 \otimes B_2)x\rangle_H \\ \langle x, (A_2 \otimes B_1)x\rangle_H & \langle x, (A_2 \otimes B_2)x\rangle_H \end{pmatrix},$$

where $A_i := R(\alpha_i)$, $B_j := R(\beta_j)$ *and*

$$\mathbb{M}(2 \times 2; \mathbb{C}) \ni R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix},$$

Then $\Gamma^{EPR} \in QC(2 \times 2; \mathbb{R})$ *and*

$$|\langle A^{Had}, \Gamma^{EPR}\rangle| = |tr(A^{Had}\Gamma^{EPR})| = 2\sqrt{2} > 2.$$