Sklar's Theorem and the Rüschendorf transform revisited - An analysis of right quantiles

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Copulas and Their Applications
TO COMMEMORATE THE 75TH BIRTHDAY OF
PROFESSOR ROGER B. NELSEN
University of Almería
July 3 - 5, 2017

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Sklar's Theorem

Let F be a n-dimensional distribution function with marginals $F_1, \ldots, F_n$. Then there exists a copula C_F, such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ we have

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If in addition F is continuous, the copula C_F is unique. Conversely, for any univariate distribution functions $H_1, \ldots, H_n$, and any copula C, the composition $C \circ (H_1, \ldots, H_n)$ defines a n-dimensional distribution function with marginals $H_1, \ldots, H_n$.

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In particular, for all $(x, \lambda) \in \mathbb{R} \times [0, 1]$ the following inequality holds:

$$F(x-) \leq F_\lambda(x) \leq F(x) \, .$$

Reminder on right- and left-quantile functions

Consider the generalised inverse function $F^{\wedge} : (0,1) \longrightarrow \mathbb{R}$, given by

$$F^{\wedge}(\alpha) := \min\{x \in \mathbb{R} : F(x) \geq \alpha\} .$$

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and

$$F\left(F^{\wedge}(\alpha)-\right) \leq \alpha \leq F\left(F^{\wedge}(\alpha)+\right) = F\left(F^{\wedge}(\alpha)\right)$$

for all $\alpha \in (0,1)$.

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$$F_V(X) := R_F(X, V) = F(X-) + V\left(F(X) - F(X-)\right)$$

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(i) $F_V(X) \sim U(0,1)$ *is a uniformly distributed random variable.*

(ii) *Moreover,*

$$X = F^{\wedge}(F_V(X)) = F^{\wedge}\big(F(X-) + V\,(F(X) - F(X-))\big) \ \mathbb{P}\text{-a.s.}$$

Rüschendorf's proof of Sklar's Theorem II

Proof of Sklar

Let $i \in \{1, 2, \ldots, n\}$ and $V_i \sim U(0, 1)$. On $\{0 < V_i \leq 1\}$ put

$$U_i := R_{F_i}(X_i, V_i) = F_i(X_i-) + V_i \Delta F_i(X_i) \,.$$

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According to Rüschendorf's theorem on the distributional transform there exist null sets $M_1, M_2, \ldots, M_n \in \mathcal{F}$, such that on $\Omega \setminus M_i \subseteq \{0 < V \leq 1\}$ $Z_i := F_i{}^{\wedge}(U_i)$ is well-defined and satisfies $X_i = Z_i$ for every $i \in \{1, 2, \ldots, n\}$. Thus, $\mathbb{P}(M) = 0$, where $M := \bigcup_{i=1}^{n} M_i$. Clearly, $X_i = Z_i$ holds on $\Omega \setminus M$ for all i "altogether".

Rüschendorf's proof of Sklar's Theorem III

Proof (ctd).

Consider the copula

$$C_F(\gamma_1, \dots, \gamma_n) := \mathbb{P}(U_1 \le \gamma_1, \dots, U_n \le \gamma_n)\,,$$

where $(\gamma_1, \dots, \gamma_n)^\top \in [0,1]^n$.

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$$\{u \in (0,1) : u \le F_i(x_i)\} = \{u \in (0,1) : F_i^\wedge(u) \le x_i\}$$

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$$A_\alpha \equiv A_\alpha[F] := \{(x,\lambda) \in \mathbb{R} \times (0,1] : R_F(x,\lambda) \leq \alpha\} .$$

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The following important specification of $A_\alpha[F]$ - strongly involving the *right*-quantile function $F^\vee$ - will help us considerably to calculate the interesting probability $\mathbb{P}\big((X, V) \in A_\alpha[F]\big)$.

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Note that in our approach F don't need to coincide with the distribution function of X!

The distribution of $F_V(X)$ II

Lemma

Let $\alpha \in (0,1)$ and F an arbitrary distribution function. Put $\xi := F^{\wedge}(\alpha)$, $\eta := F^{\vee}(\alpha)$, $q := F(\xi-)$, $\beta := \Delta F(\xi)$ and $\gamma := \Delta F(\eta)$. Then

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$$A_\alpha = \begin{cases} (-\infty,\eta] \times (0,1] & \text{if } \gamma = 0 \\ (-\infty,\eta) \times (0,1] & \text{if } \xi < \eta \text{ and } \gamma > 0 \\ (-\infty,\xi) \times (0,1] \,\dot\cup\, \{\xi\} \times \left(0, \frac{\alpha-q}{\beta}\right] & \text{if } \xi = \eta \text{ and } \gamma (= \beta) > 0 \end{cases}$$

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The proof of this lemma is built on the following accurate description of all “flat pieces” of F.

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Let $0 < \lambda \leq 1$ *and* $\alpha \in (0, 1)$*. Put* $\xi := F^{\wedge}(\alpha)$ *and* $\eta := F^{\vee}(\alpha)$*.*

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Observation

$\xi < \eta$ *if and only if there are* $x_1 \neq x_2$ *such that* $F(x_1) = \alpha = F(x_2)$.

The distribution of $F_V(X)$ IV

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$$\mathbb{P}\big(F_V(X) \leq \alpha\big) = \mathbb{P}\left(F_V(X) \leq \alpha \text{ and } V \in (0,1]\right) = \mathbb{P}\big((V, X) \in A_\alpha\big) .$$

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(i) *If $\xi < \eta$ then*

$$\mathbb{P}\big(F_V(X) \leq \alpha\big) = \mathbb{P}\big(X < \eta\big) + \mathbb{1}_{\{F=\alpha\}}(\eta)\,\mathbb{P}\big(X = \eta\big)$$

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Proposition (ctd.)

(ii) *If* $\xi = \eta$ *and* $\beta = 0$ *then*

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Observation

In any of the three cases the right side of the representation of $\mathbb{P}\big(F_V(X) \leq \alpha\big)$ *is completely independent of the choice of the random variable* $V \sim U(0,1)$*.*

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Hence, if $F = F_X$ coincides with the distribution function of X it follows that

$$\mathbb{P}(X \leq \xi) = \alpha = \mathbb{P}(X < \eta)\,.$$

In particular, if $\xi < \eta$:

$$\mathbb{P}\left(F(X) = \alpha\right) =$$

The special case $F = F_X$ I

Let $\xi < \eta$ or $\beta = 0$. Then

$$F(\xi) = \alpha = F(\eta-)\,.$$

Hence, if $F = F_X$ coincides with the distribution function of X it follows that

$$\mathbb{P}(X \leq \xi) = \alpha = \mathbb{P}(X < \eta)\,.$$

In particular, if $\xi < \eta$:

$$\mathbb{P}\left(F(X) = \alpha\right) = \mathbb{P}(X < \eta) - \mathbb{P}(X < \xi)$$

The special case $F = F_X$ I

Let $\xi < \eta$ or $\beta = 0$. Then

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In particular, if $\xi < \eta$:

$$\mathbb{P}\left(F(X) = \alpha\right) = \mathbb{P}(X < \eta) - \mathbb{P}(X < \xi) = \mathbb{P}\left(X = \xi\right)\ .$$

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In particular, if $\xi < \eta$:

$$\mathbb{P}\left(F(X) = \alpha\right) = \mathbb{P}(X < \eta) - \mathbb{P}(X < \xi) = \mathbb{P}\left(X = \xi\right)\,.$$

Corollary

Let $\alpha \in (0,1)$. Let X, V be two random variables, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $V \sim U(0,1)$ and V is independent of X. Let $F = F_X$ be the distribution function of X. Then $F_V(X)$ is uniformly distributed.

The special case $F = F_X$ II

Corollary (ctd.)

Moreover, if F has at least one "flat piece" then

$$\mathbb{P}\big(F(X) \leq \alpha\big) = \alpha = \mathbb{P}\big(X \leq F^{\wedge}(\alpha)\big)$$

on the set $\big\{\alpha \in (0,1) : F^{\wedge}(\alpha) < F^{\vee}(\alpha)\big\}$.

The special case $F = F_X$ II

Corollary (ctd.)

Moreover, if F has at least one "flat piece" then

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Observation

Let $F = F_X$ be a distribution function of a given random variable X. TFAE:

(i) *F is continuous.*

(ii) *$F(X)$ is uniformly distributed over $(0,1)$.*

The special case $F = F_X$ II

Corollary (ctd.)

Moreover, if F has at least one "flat piece" then

$$\mathbb{P}\big(F(X) \leq \alpha\big) = \alpha = \mathbb{P}\big(X \leq F^{\wedge}(\alpha)\big)$$

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Observation

Let $F = F_X$ be a distribution function of a given random variable X. TFAE:

(i) *F is continuous.*

(ii) *$F(X)$ is uniformly distributed over $(0,1)$.*

Consequently, if F had non-zero jumps $F_V(X)$ still would be uniformly distributed over $(0,1)$, as opposed to $F(X)$.

The special case $F = F_X$ III

Proof.
By contradiction assume that (ii) holds and (i) is false. Then there exists $x_0 \in \mathbb{R}$ such that

$$0 < \Delta F(x_0) = F(x_0) - F(x_0-) = \mathbb{P}(X = x_0) \leq \mathbb{P}(F(X) = F(x_0)) = 0$$

- which is a contradiction. $\square$

The special case $F = F_X$ IV

Theorem

Let X, V be two random variables, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $V \sim U(0,1)$ and V is independent of X. Let $F = F_X$ be the distribution function of the random variable X. Then

$$X = F^{\wedge}\big(F_V(X)\big) = F^{\wedge}\big(F(X-) + V\Delta F(X)\big) \quad \mathbb{P}\text{-almost surely.}$$

If in addition $\mathbb{P}\big(0 < F(X) < 1\big) = 1$ (for example, if F is continuous), then

$$X = F^{\wedge}\big(F(X)\big) \quad \mathbb{P}\text{-almost surely}.$$

The special case $F = F_X$ V

Proof (Sketch).

Since $V \sim U(0,1)$, it follows that on $\mathcal{B}\big((0,1]\big)$ the image measure $\mu_{F_V} := \mathbb{P}(V \in \cdot)$ coincides with the Lebesgue measure m.

Proof (Sketch).

Since $V \sim U(0,1)$, it follows that on $\mathcal{B}\big((0,1]\big)$ the image measure $\mu_{F_V} := \mathbb{P}(V \in \cdot)$ coincides with the Lebesgue measure m.

Thus, by applying the Fubini-Tonelli Theorem to non-negative (or bounded) Borel functions on $\big(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu_{F_V} \otimes \mu_{F_X}\big)$ we firstly recognise that there exists an m-null set $L \in \mathcal{B}\big((0,1]\big)$ such that for all $\lambda \in A := (0,1] \setminus L$ there exists a Borel set $N_\lambda \subseteq \mathbb{R}$ such that $\mathbb{P}\left(X \in N_\lambda\right) = 0$ and for any $x \in \mathbb{R} \setminus N_\lambda$ the value $F^\wedge\big(F_\lambda(x)\big)$ is well-defined and satisfies $F^\wedge\big(F_\lambda(x)\big) = x$.

The special case $F = F_X$ VI

Proof (Sketch) - ctd.

Hence, since

$$\mathbb{P}\big(X \in N_V \text{ and } V \in A\big) = \int_A \mathbb{P}\big(X \in N_\lambda\big) m(\mathrm{d}\lambda) = 0\,,$$

it consequently follows

$$X = F^\wedge\big(F_V(X)\big) = F^\wedge\big(F(X-) + V\Delta F(X)\big)$$

on the set $\Omega \setminus N \subseteq \{0 < V \leq 1\}$, where
$N := \{V \notin A\} \dot\cup \{X \in N_V \text{ and } V \in A\}$.

The special case $F = F_X$ VI

Proof (Sketch) - ctd.

Hence, since

$$\mathbb{P}\big(X \in N_V \text{ and } V \in A\big) = \int_A \mathbb{P}\big(X \in N_\lambda\big) m(\mathrm{d}\lambda) = 0\,,$$

it consequently follows

$$X = F^{\wedge}\big(F_V(X)\big) = F^{\wedge}\big(F(X-) + V\Delta F(X)\big)$$

on the set $\Omega \setminus N \subseteq \{0 < V \leq 1\}$, where
$N := \{V \notin A\} \dot\cup \{X \in N_V \text{ and } V \in A\}$.
The second part of the claim follows from

$$X(\omega) = F^{\wedge}\left(F(X(\omega)-) + V(\omega)\Delta F(X(\omega))\right) \leq F^{\wedge}(F(X(\omega))) \leq X(\omega)$$

outside of a $\mathbb{P}$-null set $\tilde{N}$, satisfying $N \subseteq \tilde{N}$. ■

Not "confusing". Rather "quite hidden" I would suggest

The work described here exploits well known but, to our knowledge, seldom exploited, differentiability properties of copulas. It has been our experience that confusing issues in stochastic processes can often be reduced to clear issues in real analysis by reformulation in terms of copulas and further that, because of the nice properties of copulas and their first partial derivatives, the resulting issues can often be addressed and answered by simple classical arguments.

- ***W. F. Darsow, B. Nguyen and E. T. Olsen (1996)***

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Thank you for your attention!

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Are there any questions, comments or remarks?