A statistical interpretation of Grothendieck's inequality: towards an approximation of the real Grothendieck constant

Frank Oertel

Philosophy, Logic & Scientific Method
Centre for Philosophy of Natural and Social Sciences (CPNSS)
London School of Economics & Political Science, UK
http://www.frank-oertel-math.de

Geometrical Aspects of Banach Spaces

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Contents

An important information for readers

The current version is a shortened ("web-publishable" ☺) version of my presentation in Birmingham.

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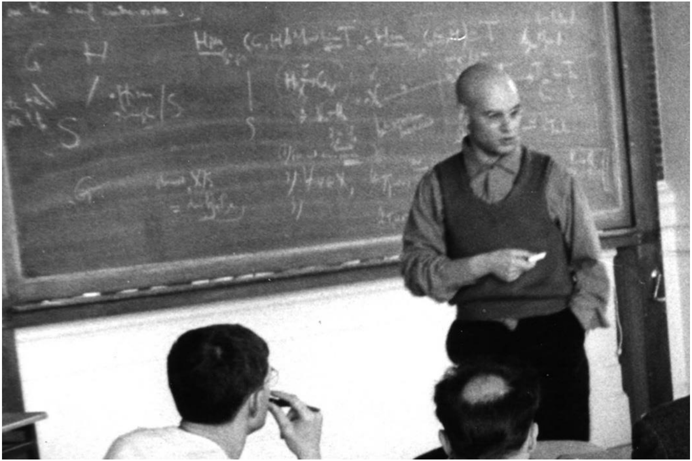
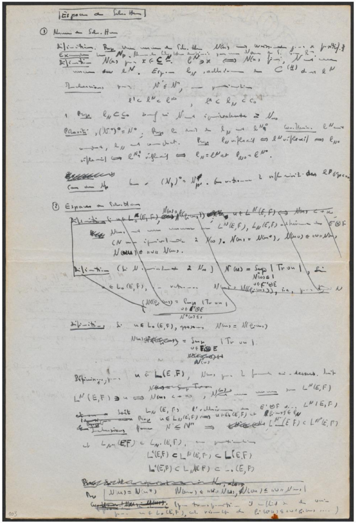
The current version is a shortened ("web-publishable" ☺) version of my presentation in Birmingham. If you are interested in a copy of my original slides, presented in Birmingham, could you please send an email to me in advance? I am happy to forward these to you, of course. Many thanks!

A portrait of A. Grothendieck

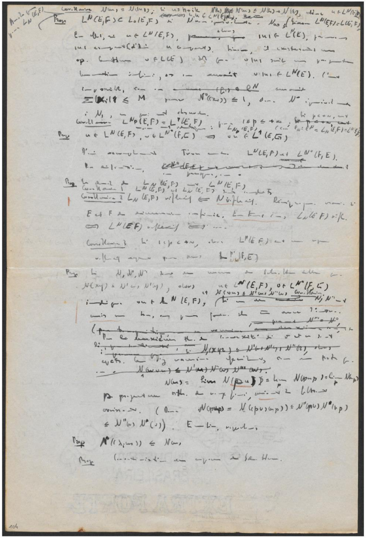
A. Grothendieck lecturing at IHES (1958-1970)



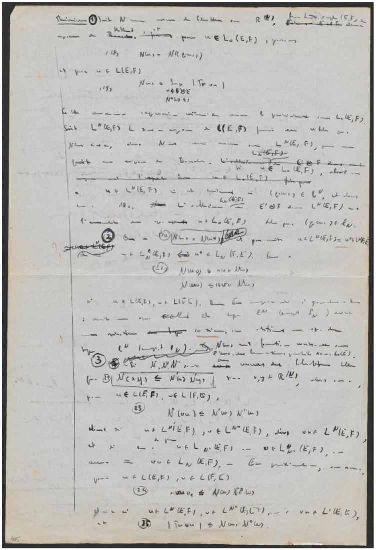
Excerpt from A. Grothendieck's handwritten lecture notes I



Excerpt from A. Grothendieck's handwritten lecture notes II



Excerpt from A. Grothendieck's handwritten lecture notes III



Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $m, n \in \mathbb{N}$. *Then there exists a universal constant* $K > 0$ *- not depending on* m *and* n *- such that for all matrices* $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$, *for all* $\mathbb{F}$*-Hilbert spaces* H, *for all unit vectors* $u_1, \ldots, u_m, v_1, \ldots, v_n \in S_H$ *the following inequality is satisfied:*

$$\Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K \max \Big\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| : p_i, q_j \in \{-1, 1\} \Big\} .$$

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant.

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant. Computing the exact numerical value of this constant is an open problem (unsolved since 1953)!

Grothendieck's inequality in matrix form II

Theorem (R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and H be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$. Let $K_{GH}^{\mathbb{F}}$ denote the Grothendieck constant, derived from Grothendieck's inequality "restricted" to the set of all positive semidefinite $n \times n$-matrices over $\mathbb{F}$. Then

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$$K_{GH}^{\mathbb{R}} = \frac{\pi}{2}$$

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$$K_{GH}^{\mathbb{R}} = \frac{\pi}{2} \quad \text{and} \quad K_{GH}^{\mathbb{C}} = \frac{4}{\pi}.$$

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$$K_{GH}^{\mathbb{R}} = \frac{\pi}{2} \quad \text{and} \quad K_{GH}^{\mathbb{C}} = \frac{4}{\pi}\,.$$

From now on are going to consider the real case (i. e., $\mathbb{F} = \mathbb{R}$) only. Nevertheless, we allow an unrestricted use of all matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ for any $m, n \in \mathbb{N}$ in GT.

Grothendieck's inequality in matrix form III

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily thanks to R. E. Rietz (1974), J. L. Krivine (1977), and recently, due to an impressive work of M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (2011)*:

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$$1,676 < K_G^{\mathbb{R}} \overset{(!!)}{<} \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1,782\,.$$

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Conjecture

Is $K_G^{\mathbb{R}} = \sqrt{\pi}$

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Is $K_G^{\mathbb{R}} = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$

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Is $K_G^{\mathbb{R}} = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) \approx 1,772$*?*

Grothendieck's inequality rewritten I

By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear mappings between correlation matrices.

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By transforming Grothendieck's inequality into an equivalent inequality between traces of matrix products (respectively Hilbert-Schmidt inner products) we are lead to a surprising interpretation which reveals deep links to combinatorial (binary) optimisation, semidefinite programming (SDP) and multivariate statistics, built on suitable non-linear mappings between correlation matrices.

We will sketch this approach which might lead to a constructive improvement of Krivine's upper bound $\frac{\pi}{2\ln(1+\sqrt{2})}$. At least it also can be reproduced in this approach.

Grothendieck's inequality rewritten II

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

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Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H =$$

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$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \text{tr}\big(A^\top \, \Gamma_H(u, v)\big) =$$

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is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{R})$, where

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$$\Gamma_H(u, v) := \begin{pmatrix} \langle u_1, v_1 \rangle_H & \langle u_1, v_2 \rangle_H & \ldots & \langle u_1, v_n \rangle_H \\ \langle u_2, v_1 \rangle_H & \langle u_2, v_2 \rangle_H & \ldots & \langle u_2, v_n \rangle_H \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1 \rangle_H & \langle u_m, v_2 \rangle_H & \ldots & \langle u_m, v_n \rangle_H \end{pmatrix}.$$

Grothendieck's inequality rewritten III

Let $m, n \in \mathbb{N}, A \in \mathbb{M}(m \times n; \mathbb{R}), p := (p_1, \ldots, p_m)^\top \in \left(\mathbb{S}^0\right)^m$ and $q := (q_1, \ldots, q_n)^\top \in \left(\mathbb{S}^0\right)^n$ be given, where $\mathbb{S}^0 := \{-1, 1\}$ denotes the unit "sphere" in $\mathbb{R} = \mathbb{R}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \operatorname{tr}\big(A^\top \Gamma_{\mathbb{R}}(p, q)\big) = \langle A, \Gamma_{\mathbb{R}}(p, q) \rangle,$$

where now

$$\Gamma_{\mathbb{R}}(p, q) := pq^\top = \begin{pmatrix} \pm 1 & \mp 1 & \ldots & \pm 1 \\ \mp 1 & \mp 1 & \ldots & \mp 1 \\ \vdots & \vdots & \vdots & \vdots \\ \pm 1 & \mp 1 & \ldots & \pm 1 \end{pmatrix}.$$

Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

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Does this matrix look familiar to you?

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It is a part of something larger...

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Namely:

Block matrix representation I

$$\begin{pmatrix} & \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix} \\ \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}^{\mathbf{T}} & \end{pmatrix}$$

Block matrix representation I

$$
\begin{pmatrix}
 & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \ldots & \langle u_1, v_n\rangle \\
 & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \ldots & \langle u_2, v_n\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \ldots & \langle u_m, v_n\rangle \\
\langle u_1, v_1\rangle & \langle u_2, v_1\rangle & \ldots & \langle u_m, v_1\rangle & & & & \\
\langle u_1, v_2\rangle & \langle u_2, v_2\rangle & \ldots & \langle u_m, v_2\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle u_1, v_n\rangle & \langle u_2, v_n\rangle & \ldots & \langle u_m, v_n\rangle & & & &
\end{pmatrix}
$$

Block matrix representation I

$$
\begin{pmatrix}
 & & & & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
 & & & & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & & & & \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & & & &
\end{pmatrix}
$$

Block matrix representation II

$$\begin{pmatrix}
\langle u_1, u_1\rangle & \langle u_1, u_2\rangle & \dots & \langle u_1, u_m\rangle & \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\
\langle u_2, u_1\rangle & \langle u_2, u_2\rangle & \dots & \langle u_2, u_m\rangle & \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1\rangle & \langle u_m, u_2\rangle & \dots & \langle u_m, u_m\rangle & \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \\
\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & \dots & \langle v_1, u_m\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\
\langle v_2, u_1\rangle & \langle v_2, u_2\rangle & \dots & \langle v_2, u_m\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1\rangle & \langle v_n, u_2\rangle & \dots & \langle v_n, u_m\rangle & \langle v_n, v_1\rangle & \langle v_m, v_2\rangle & \dots & \langle v_n, v_n\rangle
\end{pmatrix}$$

Block matrix representation III

$$\begin{pmatrix}
1 & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_m \rangle & \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \dots & \langle u_1, v_n \rangle \\
\langle u_2, u_1 \rangle & 1 & \dots & \langle u_2, u_m \rangle & \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \dots & \langle u_2, v_n \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle u_m, u_1 \rangle & \langle u_m, u_2 \rangle & \dots & 1 & \langle u_m, v_1 \rangle & \langle u_m, v_2 \rangle & \dots & \langle u_m, v_n \rangle \\
\langle v_1, u_1 \rangle & \langle v_1, u_2 \rangle & \dots & \langle v_1, u_m \rangle & 1 & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_n \rangle \\
\langle v_2, u_1 \rangle & \langle v_2, u_2 \rangle & \dots & \langle v_2, u_m \rangle & \langle v_2, v_1 \rangle & 1 & \dots & \langle v_2, v_n \rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle v_n, u_1 \rangle & \langle v_n, u_2 \rangle & \dots & \langle v_n, u_m \rangle & \langle v_n, v_1 \rangle & \langle v_m, v_2 \rangle & \dots & 1
\end{pmatrix}$$

A refresher of a few definitions I

Let $n \in \mathbb{N}$. We put

$PSD(n;\mathbb{R}) := \{S : S \in \mathbb{M}(n \times n;\mathbb{R})$ and S is positive semidefinite$\}$.

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Let $n \in \mathbb{N}$. We put

$PSD(n;\mathbb{R}) := \{S : S \in \mathbb{M}(n \times n;\mathbb{R}) \text{ and } S \text{ is positive semidefinite}\}$.

Recall that $PSD(n;\mathbb{R})$ is a closed convex cone which (by definition!) consists of symmetric matrices only.

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Moreover, we consider the set

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Let $d, k \in \mathbb{N}$ and $(H, \langle \cdot, \cdot \rangle)$ be an arbitrary d-dimensional Hilbert space (i. e, $H = l_2^d$). Let $w_1, w_2, \ldots, w_k \in H$. Put $w := (w_1, \ldots, w_k)^\top \in H^k$ and $S := \left(w_1 \,\vdots\, w_2 \,\vdots\, \ldots \,\vdots\, w_k\right) \in \mathbb{M}(d \times k; \mathbb{R})$. The matrix $\Gamma_H(w, w) \in PSD(k; \mathbb{R})$, defined as

$$\Gamma_H(w, w)_{ij} := \langle w_i, w_j \rangle = \left(S^\top S\right)_{ij} \qquad \left(i, j \in [k] := \{1, 2, \ldots, k\}\right)$$

is called Gram matrix of the vectors $w_1, \ldots, w_k \in H$.

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Observe that

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is not a Gram matrix!

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Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

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Recall that ξ is an n-dimensional Gaussian random vector with respect to the "parameters" μ and C (short: $\xi \sim N_n(\mu, C)$) if and only if for all $a \in \mathbb{R}^n$ there exists $\eta_a \sim N_1(0, 1)$ such that

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Note that we don't require here that C is invertible! Following Feller, the matrix $\mathbb{V}(\xi)$ defined as

$$\mathbb{V}(\xi)_{ij} := \mathbb{E}[\xi_i \xi_j] - \mathbb{E}[\xi_i]\mathbb{E}[\xi_j] \stackrel{(!)}{=} C_{ij} \qquad (i, j \in [n])$$

is known as the *variance matrix of the Gaussian random vector* ξ.

Structure of correlation matrices I

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In particular, condition (i) implies that $\sigma_{ij} \in [-1, 1]$ *for all* $i, j \in [n]$.

Structure of correlation matrices II

Observation

Let $k \in \mathbb{N}$. Then the sets $\left\{S : S = xx^\top \text{ for some } x \in \{-1, 1\}^k\right\}$ and $\left\{\Theta : \Theta \in C(k; \mathbb{R}) \text{ and } rk(\Theta) = 1\right\}$ coincide.

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Proposition (K. R. Parthasarathy (2002))

Let $k \in \mathbb{N}$. $C(k;\mathbb{R})$ is a compact and convex subset of the k^2-dimensional vector space $\mathbb{M}(k \times k;\mathbb{R})$. Any $k \times k$-correlation matrix of rank 1 is an extremal point of the set $C(k;\mathbb{R})$.

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Let $k \in \mathbb{N}$. Put

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Having gained - important - additional structure by “enlarging” the $m \times n$-matrix $\Gamma_H(u, v)$ to a $(m+n) \times (m+n)$-correlation matrix, how could this gained information be used to rewrite Grothendieck’s inequality accordingly?

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Definition

Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{R})$ arbitrary. Put

$$\widehat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

Let us call $\mathbb{M}((m+n) \times (m+n); \mathbb{R}) \ni \widehat{A}$ the canonical block injection of A.

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Observe that $\widehat{A}$ is symmetric, implying that $\widehat{A} = \widehat{A}^\top$.

A further equivalent rewriting of GT I

Proposition

Let $m, n \in \mathbb{N}$ *and* $A = \left(a_{ij}\right) \in \mathbb{M}(m \times n; \mathbb{R})$. *Let* $K > 0$. *TFAE:*

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We don't know whether condition (ii) holds for all matrices in $\mathbb{M}((m+n) \times (m+n); \mathbb{R})$.

GT versus NP-hard optimisation

Observation

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

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Thus, Grothendieck's constant $K_G^{\mathbb{R}}$ is precisely the "integrality gap"; i. e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!

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Let $\emptyset \neq U \subseteq \mathbb{R}$ and $f : U \longrightarrow \mathbb{R}$ a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{R})$ such that $a_{ij} \in U$ for all $(i, j) \in [m] \times [n]$.

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Guiding Example

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The notation "$f[A]$" is used to highlight the difference between the matrix $f(A)$ originating from the spectral representation of A (for normal matrices A) and the matrix $f[A]$, defined as above !

Schur product and the matrix $f[A]$ II

Theorem

For any $n \in \mathbb{N}$ the set $\mathbb{M}(n \times n; \mathbb{R})$ with the usual addition and the Schur multiplication $*$ is a commutative Banach algebra under the operator norm. Moreover,

$$PSD(n;\mathbb{R}) * PSD(n;\mathbb{R}) \subseteq PSD(n;\mathbb{R}) \, .$$ [1]

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Let $0 < r \leq \infty$ *and* $f : [-r, r] \longrightarrow \mathbb{R}$ *such that on* $[-r, r]$

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$$f[A] \in PSD(n; [-f(r), f(r)]) \text{ for all } n \in \mathbb{N} \text{ and } A \in PSD(n; [-r, r]) .$$

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$$f[A] \in PSD(n; [-f(r), f(r)]) \text{ for all } n \in \mathbb{N} \text{ and } A \in PSD(n; [-r, r]) .$$

If in addition $r \geq 1$ *and* $f \neq 0$ *then* $f(1) > 0$, *and*

$$\frac{f}{f(1)}[\cdot] : C(n, \mathbb{R}) \longrightarrow C(n, \mathbb{R}) \text{ for every } n \in \mathbb{N} .$$

Correlation matrix transformation II

Example

Let $x \in [-1, 1]$.

$$g(x) \quad := \quad \frac{2}{\pi}\arcsin(x) \; = \; \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}$$

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where for $a, b, c \in \mathbb{C}$*, satisfying* $\Re\big(c-(a+b)\big) > 0$ *and* $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$${}_2F_1\left(a,b,c;z\right) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\,n!}\,z^n$$

denotes the hypergeometric function.

Correlation matrix transformation III

Example

Let H be an arbitrary separable Hilbert space with ONB $(x_n)_{n\in\mathbb{N}_0}$. Let $\psi \in H$ such that $\|\psi\| > 0$.

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Then $\dfrac{h_\psi}{\|\psi\|^2}[\cdot] : C(k,\mathbb{R}) \longrightarrow C(k,\mathbb{R})$ *for every* $k \in \mathbb{N}$.

Correlation matrix transformation IV

Theorem

Let $1 \leq r \leq \infty$. Let $g : [-r, r] \longrightarrow \mathbb{R}$ be a function such that for each $n \in \mathbb{N}$ $g\big[\Sigma\big]$ is a $(n \times n)$-correlation matrix for all $(n \times n)$-correlation matrices Σ.

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Moreover, $g[A] \in PSD(n; \mathbb{R})$ *for all* $A \in PSD(n; [-r, r])$ *and all* $n \in \mathbb{N}$*.*

Correlation matrix transformation and Schoenberg's Theorem

Even more holds:

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Theorem (Schoenberg (1942), Rudin (1959), Guillot and Rajaratnam (2012))

Let $0 < \alpha \leq \infty$ *and* $g : (-\alpha, \alpha) \longrightarrow \mathbb{R}$ *be an arbitrary function.*
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(ii) g *is a smooth function that admits a power series representation* $g(x) = \sum_{k=0}^{\infty} b_k x^k$ *on* $(-\alpha, \alpha)$ *for some sequence* (b_n)*, consisting of nonnegative numbers only.*

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A: Apply "suitable" correlation matrix transforms!

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Theorem (Grothendieck's identity - T. S. Stieltjes (1889), W. F. Sheppard (1898))

Let $-1 \leq t \leq 1$ *and* $(\xi, \eta)^\top \sim N_2(0, \Sigma_t)$, *where*

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Corollary

Let $k \in \mathbb{N}, k \geq 2$*. Let* $\Sigma \in C(k;\mathbb{R})$ *an arbitrarily given correlation matrix.*

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Let $k \in \mathbb{N}, k \geq 2$. *Let* $\Sigma \in C(k;\mathbb{R})$ *an arbitrarily given correlation matrix. Then also* $\frac{2}{\pi}\arcsin[\Sigma] \in C(k;\mathbb{R})$.

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Let $k \in \mathbb{N}, k \geq 2$. *Let* $\Sigma \in C(k;\mathbb{R})$ *an arbitrarily given correlation matrix. Then also* $\frac{2}{\pi}\arcsin[\Sigma] \in C(k;\mathbb{R})$. *There exists a Gaussian random vector* $\xi \sim N_k(0,\Sigma)$ *such that*

$$\frac{2}{\pi}\arcsin[\Sigma] = \mathbb{E}\left[\Theta(\xi)\right],$$

where

$$\Theta(\xi(\omega))_{ij} := \textit{sign}(\xi_i(\omega))\textit{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$, *and for all* $i,j \in [k]$.

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for all $\omega \in \Omega$, and for all $i,j \in [k]$. $\Theta(\xi(\omega))$ is a correlation matrix of rank 1 for all $\omega \in \Omega$, and we have

$$\max_{\substack{\Theta \in C(k;\mathbb{R}) \\ rank(\Theta)=1}} |\langle \widehat{A}, \Theta\rangle| \geq \mathbb{E}\big[|\langle \widehat{A}, \Theta(\xi)\rangle|\big] \geq |\langle \widehat{A}, \mathbb{E}\big[\Theta(\xi)\big]\rangle| = \frac{2}{\pi}|\langle \widehat{A}, \arcsin[\Sigma]\rangle|\,.$$

Krivine's constant reproduced I

Example

$$f(x) := \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{R})$$

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A bit more generally, the following (still special) case holds:

Krivine's constant reproduced II

Corollary

Let $m, n \in \mathbb{N}$*,* $A \in \mathbb{M}(m \times n; \mathbb{R})$*,* $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ *and* $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$*. Put* $S := \Gamma_H(u, v)$*. Let* $0 < r \leq \infty$ *and* $f : (-r, r) \longrightarrow \mathbb{R}$ *be a function such that* f *satisfies the "correlation assumptions".*

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Let $m, n \in \mathbb{N}$, $A \in \mathbb{M}(m \times n; \mathbb{R})$, $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ *and* $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$. *Put* $S := \Gamma_H(u, v)$. *Let* $0 < r \leq \infty$ *and* $f : (-r, r) \longrightarrow \mathbb{R}$ *be a function such that* f *satisfies the "correlation assumptions". Assume that* $f(c^*) = 1$ *for some* $0 < c^* < r$. *Then*

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$$\max_{\Theta \in C_1(m+n;\mathbb{R})} \left| \langle \widehat{A}, \Theta \rangle \right|$$

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$$\max_{\Theta \in C_1(m+n;\mathbb{R})} \left| \langle \widehat{A}, \Theta \rangle \right| \quad \geq \quad \frac{2}{\pi} \left| tr\big(A^\top (c^* S)\big) \right|$$

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Corollary

Let $m, n \in \mathbb{N}$, $A \in \mathbb{M}(m \times n; \mathbb{R})$, $u := (u_1, u_2, \ldots, u_m)^\top \in S_H^m$ *and* $v := (v_1, v_2, \ldots, v_n)^\top \in S_H^n$. *Put* $S := \Gamma_H(u, v)$. *Let* $0 < r \leq \infty$ *and* $f : (-r, r) \longrightarrow \mathbb{R}$ *be a function such that* f *satisfies the "correlation assumptions". Assume that* $f(c^*) = 1$ *for some* $0 < c^* < r$. *Then*

$$\max_{\Theta \in C_1(m+n;\mathbb{R})} \left| \langle \widehat{A}, \Theta \rangle \right| \geq \frac{2}{\pi} \left| tr\big(A^\top (c^* S)\big) \right| = \frac{2c^*}{\pi} \left| tr\big(A^\top \Gamma_H(u, v)\big) \right|.$$

Krivine's constant reproduced II

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Hence,

$$\boxed{K_G^{\mathbb{R}} \leq \frac{\pi}{2\, c^*}}.$$

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Somewhat more generally, we have

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Krivine rounding schemes revisited I

Natural Question

Can we possibly "remove" or cleverly substitute the arcsin *function?*

Krivine rounding schemes revisited I

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Can we possibly "remove" or cleverly substitute the $\arcsin$ *function?*

Recall that the invertible function $\frac{2}{\pi}\arcsin : [-1,1] \longrightarrow [-1,1]$ transforming correlation matrices into correlation matrices arrived as a result of an explicit hard calculation of the *non-trivial double integral*

$$H_{f,g}(t) := \mathbb{E}[f(\xi)\, g(\eta)]\,,$$

where $\xi, \eta \sim N(0,1)$ are correlated via $\mathbb{E}[\xi\,\eta] = t \in [-1,1]$ and $f := g :=$ sign $= \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0)}$.

Krivine rounding schemes revisited II

Definition (Braverman, Makarychev, Makarychev, Naor (2011))

Fix $k \in \mathbb{N}$. Let $G_1, G_2 \sim N_k(0, I_k)$ be independent (standard) Gaussian random vectors. Let $t \in [-1, 1]$. Put $\tilde{f}(x) := f(\frac{1}{\sqrt{2}}\, x)$ and $\tilde{g}(y) := g(\frac{1}{\sqrt{2}}\, y)$, where $x, y \in \mathbb{R}^k$ and f, g are bounded.

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$$H_{f,g}(t) \quad := \quad \mathbb{E}\big[\tilde{f}(G_1)\tilde{g}(tG_1 + \sqrt{1 - t^2}\, G_2)\big]$$

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&\overset{(!)}{=} \int_{\mathbb{R}^k} \tilde{f}(x)\mathbb{E}\big[\tilde{g}(tx + \sqrt{1 - t^2}\, G_2)\big]\gamma_k(dx)
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Krivine rounding schemes revisited II

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A lurking Mehler kernel

Observation

Fix $k \in \mathbb{N}$*. Let* $G \sim N_k(0, I_k)$ *be a (standard) Gaussian random vector. Let* $-1 < t < 1$ *and* g *be as above.*

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$$M_t(x,y) \;:=\; \frac{1}{(1-t^2)^{k/2}} \exp\Big(- \frac{t^2\big(\|x\|^2 + \|y\|^2\big) - 2t\langle x, y\rangle}{2(1-t^2)} \Big)$$

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A bird's eye view - from the complex plane I

For the moment *assume that in additon* $f(-x) = -f(x)$ *for all* $x \in \mathbb{R}^k$ *or* $g(-y) = -g(y)$ *for all* $y \in \mathbb{R}^k$. Having Fourier transform techniques in mind, let us assume further that the function $z \mapsto H_{f,g}(z)$ can be analytically extended to a suitable domain in $\mathbb{C}$, containing $\pm i$.

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where

A bird's eye view - from the complex plane II

$$K(x,y) \quad := \quad \sin\left(\langle x,y\rangle\right)\varphi_{0,I_{2k}}(x,y)$$

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A similar non-trivial integral was explicitly calculated by H. König in 2001, leading to his conjecture whether $K_G^{\mathbb{R}} = \frac{\pi}{2\ln(1+\sqrt{2})}$ - which had been refuted in 2011 only (cf. [1]) !

Krivine rounding schemes revisited III

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$$\mathbb{R}^k \ni x \mapsto \chi_\alpha(x) := 1 - 2 \prod_{i=1}^{k} \mathbb{1}_{\left(-\infty, \frac{\Phi^{-1}(\alpha_i)}{\sqrt{2}}\right]}\left(x_i\right) .$$

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Then $\chi_\alpha(x) \in \{-1, 1\}$ *for all* $x \in \mathbb{R}^k$, *and*

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Then $\chi_\alpha(x) \in \{-1, 1\}$ *for all* $x \in \mathbb{R}^k$, *and*

$$H_{\chi_\alpha, \chi_\beta}(t) \overset{(!)}{=} 1 - 2\Big(\prod_{i=1}^{k} \alpha_i + \prod_{i=1}^{k} \beta_i\Big) + 4 c_{\Sigma(t)}(\alpha, \beta) \, ,$$

where $c_{\Sigma(t)}$ *denotes the* $2k$*-dimensional Gaussian copula with respect to the correlation matrix* $\Sigma(t)$*!*

Krivine rounding schemes revisited IV

A special case is the following (meanwhile not unknown !)

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$$= \frac{2}{\pi}\arcsin(t)$$

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Krivine rounding schemes revisited IV

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Krivine rounding schemes revisited V

Theorem

Let $l_2 \cong H$ *be a separable Hilbert space,* $m, n \in \mathbb{N}$*,* $k := m + n$ *and* $f, g : \mathbb{R}^k \longrightarrow \mathbb{R}$ *be bounded. Suppose the following conditions are satisfied:*

Krivine rounding schemes revisited V

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(i) $H'_{f,g}(0) \neq 0$ *and* $H_{f,g}(0) \geq 0$*.*

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Then for all $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in S_H$ *there exist* k $\mathbb{R}^k$*-valued random vectors* $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ *such that* $X_i \sim N_k(0, I_k)$ *for all* $i \in [m]$ *and* $Y_j \sim N_k(0, I_k)$ *for all* $j \in [n]$ *and*

$$\frac{1}{c}\,\mathbb{E}\Big[f\Big(\frac{1}{\sqrt{2}}X_i\Big)\,g\Big(\frac{1}{\sqrt{2}}Y_j\Big)\Big] = \langle u_i, v_j\rangle_H$$

for all $(i, j) \in [m] \times [n]$.

Krivine rounding schemes revisited V

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for all $(i, j) \in [m] \times [n]$. *If* $f, g : \mathbb{R}^k \longrightarrow \{-1, +1\}$ *then* $K_G^{\mathbb{R}} \leq \frac{1}{c(f,g)}$.

A phrase of G. H. Hardy

"... at present I will say only that if a chess problem is, in the crude sense, 'useless', then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects..."

– *A Mathematician's Apology* (1940)

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Thank you for your attention!

Are there any questions, comments or remarks?