Completely correlation preserving mappings: a way to tackle the upper bound of the real and complex Grothendieck constant?

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Work In Progress

MCMP - Munich Center for Mathematical Philosophy
LMU - Ludwig-Maximilians-Universität München

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6. Krivine's upper bound and beyond

An important information for readers

The current version is a shortened ("web-publishable" ☺) version of my presentation at the MCMP in Munich.

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The current version is a shortened ("web-publishable" ☺) version of my presentation at the MCMP in Munich. If you are interested in a copy of my original slides, presented in Munich, could you please send an email to me in advance? I am happy to forward these to you, of course. Many thanks!

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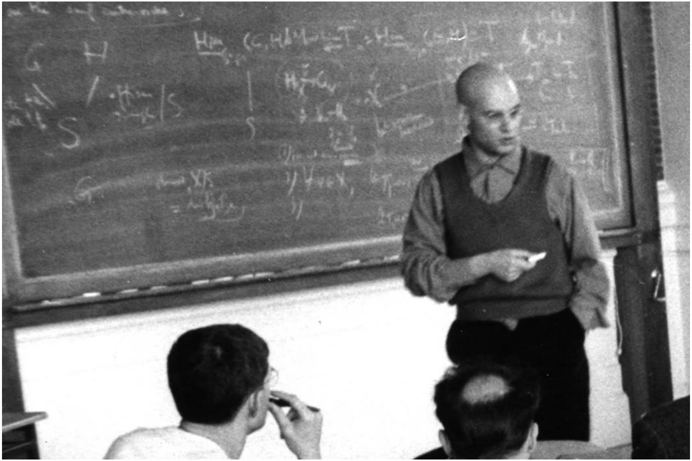
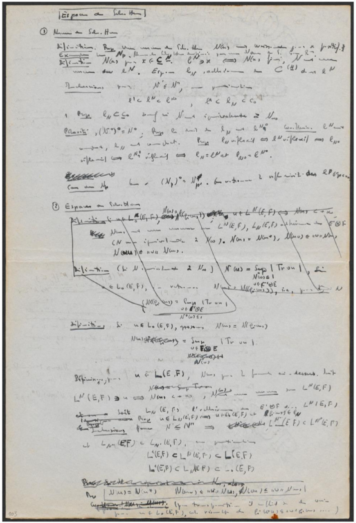
A portrait of A. Grothendieck

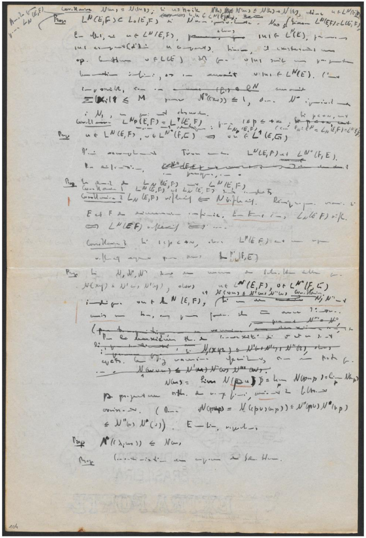
A. Grothendieck lecturing at IHES (1958-1970)



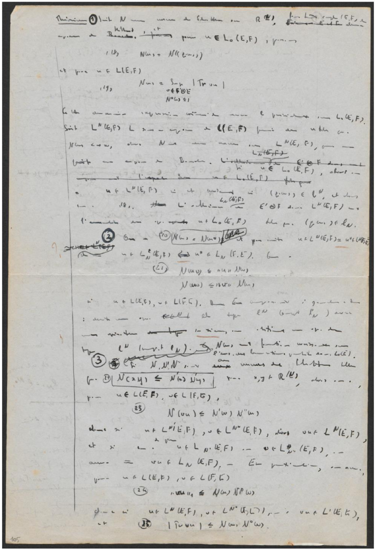
Excerpt from A. Grothendieck's handwritten lecture notes I



Excerpt from A. Grothendieck's handwritten lecture notes II



Excerpt from A. Grothendieck's handwritten lecture notes III



Grothendieck's inequality in matrix form I

Theorem (Lindenstrauss-Pelczyński (1968))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $m, n \in \mathbb{N}$. Then there exists a universal constant $K > 0$ - not depending on m and n - such that for all matrices $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$, for all $\mathbb{F}$-Hilbert spaces H, for all unit vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in S_H$ the following inequality is satisfied:

$$\Big|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle u_i, v_j\rangle_H\Big| \leq K \,\max\Big\{\Big|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} p_i q_j\Big| : |p_i| = 1 = |q_j|\Big\}.$$

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant.

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The smallest possible value of the corresponding constant K is denoted by $K_G^{\mathbb{F}}$. It is called Grothendieck's constant. Computing the exact numerical value of this constant is an open problem (unsolved since 1953)!

Grothendieck's inequality in matrix form II

Theorem ("Little Grothendieck Inequality" – R. E. Rietz (1974), H. Niemi (1983))

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and H be an arbitrary Hilbert space over $\mathbb{F}$. Let $n \in \mathbb{N}$. Let $k_G^{\mathbb{F}}$ denote the Grothendieck constant, derived from Grothendieck's inequality "restricted" to the set of all positive semidefinite $n \times n$-matrices over $\mathbb{F}$. Then

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$$k_G^{\mathbb{R}} = \frac{\pi}{2} \text{ and } k_G^{\mathbb{C}} = \frac{4}{\pi}\,.$$

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$$k_G^{\mathbb{R}} = \frac{\pi}{2} \text{ and } k_G^{\mathbb{C}} = \frac{4}{\pi}.$$

From now on we are *primarily* shedding some light on the real case in this talk (i. e., $\mathbb{F} = \mathbb{R}$). However, we allow the use of all matrices $A \in \mathbb{M}(m \times n; \mathbb{R})$ of any size, and we analyse thoroughly first non-trivial parts of the complex case.

Encapsulation of $K_G^{\mathbb{R}}$ - most recent results

Until present the following encapsulation of $K_G^{\mathbb{R}}$ holds, *primarily thanks to R. E. Rietz (1974), J. L. Krivine (1977), and recently, due to an impressive work of M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor (2011)*:

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Is $K_G^{\mathbb{R}} = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) \approx 1,772$*?*

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Brave Conjecture

Is $K_G^{\mathbb{R}} = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) \approx 1,772$*? Or is it rather* $K_G^{\mathbb{R}} = \frac{1}{\gamma} \approx 1,732$*, where* $\gamma = \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n} = -\Gamma'(1) \approx 0,57721566490153\ldots$ *denotes the Euler-Mascheroni constant...?*

The left side of Grothendieck's inequality rewritten

Let $m, n \in \mathbb{N}$, $A \in \mathbb{M}(m \times n; \mathbb{F})$, $u := (u_1, \ldots, u_m)^\top \in S_H^m$ and $v := (v_1, \ldots, v_n)^\top \in S_H^n$ be given, where $S_H := \{w \in H : \|w\| = 1\}$ denotes the unit sphere in H.

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Firstly, note that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H =$$

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$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H = \text{tr}\left(\Gamma_H(u, v)^* A\right) = \langle A, \Gamma_H(u, v) \rangle,$$

is precisely the Hilbert-Schmidt inner product (or the Frobenius inner product) of the matrices $A \in \mathbb{M}(m \times n; \mathbb{F})$ and $\Gamma_H(u, v) \in \mathbb{M}(m \times n; \mathbb{F})$, where

$$\Gamma_H(u, v) := \begin{pmatrix} \langle v_1, u_1 \rangle_H & \langle v_2, u_1 \rangle_H & \ldots & \langle v_n, u_1 \rangle_H \\ \langle v_1, u_2 \rangle_H & \langle v_2, u_2 \rangle_H & \ldots & \langle v_n, u_2 \rangle_H \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_1, u_m \rangle_H & \langle v_2, u_m \rangle_H & \ldots & \langle v_n, u_m \rangle_H \end{pmatrix}.$$

The right side of Grothendieck's inequality rewritten

Let $m, n \in \mathbb{N}$, $A \in \mathbb{M}(m \times n; \mathbb{F})$, $p := (p_1, \ldots, p_m)^\top \in (\mathbb{S}^0)^m$ and $q := (q_1, \ldots, q_n)^\top \in (\mathbb{S}^0)^n$ be given, where $\mathbb{S}^0 := \{z \in \mathbb{F} : |z| = 1\}$ denotes the unit "sphere" in $\mathbb{F} = \mathbb{F}^{0+1}$.

Similarly as before, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \text{tr}\left(\Gamma_{\mathbb{F}}(p, q)^* A\right) = \langle A, \Gamma_{\mathbb{F}}(p, q) \rangle,$$

where now $\Gamma_{\mathbb{F}}(p, q) := q p^*$.

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where now $\Gamma_{\mathbb{F}}(p,q) := qp^*$.

In the real case (i. e., if $\mathbb{F} = \mathbb{R}$) we have

$$\Gamma_{\mathbb{R}}(p,q) = \begin{pmatrix} \pm 1 & \mp 1 & \ldots & \pm 1 \\ \mp 1 & \mp 1 & \ldots & \mp 1 \\ \vdots & \vdots & \vdots & \vdots \\ \pm 1 & \mp 1 & \ldots & \pm 1 \end{pmatrix}.$$

A lurking curse of dimensionality in Grothendieck's inequality? I

A Natural Question

*Suppose, we have an urn which contains $m \cdot n$ white balls and $m \cdot n$ black balls; hence $2 \cdot m \cdot n$ white and black balls altogether. How many possibilities do we have to place $m \cdot n$ balls from this urn, with replacement, in a box consisting of $m \cdot n$ empty cells, such that after the realisation of such a placement in **any** of these cells there will be **exactly one** ball?*

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Answer

There are $2^{mn} = (\exp(m\ln(2)))^n > (\exp(0.65\, m))^n$ possibilities (a simple proof by induction shows you why). Already a small real matrix $\Gamma_{\mathbb{R}}(p, q)$ consisting of $m = 5$ rows and $n = 5$ columns allows 33'554'432 different versions; each one consisting solely of mn numbers $\varepsilon_{ij} \in \{-1, 1\}$!

A lurking curse of dimensionality in Grothendieck's inequality? II

Theorem (J. M. Hendrickx and A. Olshevsky (2010))

Unless $P = NP$ there is no polynomial time algorithm which, given a real matrix A with entries in $\{-1, 0, 1\}$, approximates $\max\left\{\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j\right| : p_i, q_j \in \{-1, 1\}\right\}$ to some fixed error with running time polynomial in the dimensions of the matrix.

Full matrix representation of the Hilbert space vectors

Pick all $m + n$ Hilbert space unit vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in H$ and represent them as

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Does this matrix look familiar to you?

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Namely:

Block matrix representation I

$$
\begin{pmatrix}
 & \begin{pmatrix} \langle v_1, u_1\rangle & \langle v_2, u_1\rangle & \dots & \langle v_n, u_1\rangle \\ \langle v_1, u_2\rangle & \langle v_2, u_2\rangle & \dots & \langle v_n, u_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_1, u_m\rangle & \langle v_2, u_m\rangle & \dots & \langle v_n, u_m\rangle \end{pmatrix} \\
\begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \dots & \langle u_1, v_n\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \dots & \langle u_2, v_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle u_m, v_1\rangle & \langle u_m, v_2\rangle & \dots & \langle u_m, v_n\rangle \end{pmatrix}^{\mathbf{T}} &
\end{pmatrix}
$$

Block matrix representation I

$$
\begin{pmatrix}
 & & & & \langle v_1, u_1\rangle & \langle v_2, u_1\rangle & \dots & \langle v_n, u_1\rangle \\
 & & & & \langle v_1, u_2\rangle & \langle v_2, u_2\rangle & \dots & \langle v_n, u_2\rangle \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & \langle v_1, u_m\rangle & \langle v_2, u_m\rangle & \dots & \langle v_n, u_m\rangle \\
\langle u_1, v_1\rangle & \langle u_2, v_1\rangle & \dots & \langle u_m, v_1\rangle & & & & \\
\langle u_1, v_2\rangle & \langle u_2, v_2\rangle & \dots & \langle u_m, v_2\rangle & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\langle u_1, v_n\rangle & \langle u_2, v_n\rangle & \dots & \langle u_m, v_n\rangle & & & &
\end{pmatrix}
$$

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$$\begin{pmatrix}
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\langle u_1, u_m\rangle & \langle u_2, u_m\rangle & \dots & \langle u_m, u_m\rangle & \langle v_1, u_m\rangle & \langle v_2, u_m\rangle & \dots & \langle v_n, u_m\rangle \\
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\end{pmatrix}$$

Block matrix representation III

$$\begin{pmatrix}
\mathbf{1} & \langle u_2, u_1\rangle & \dots & \langle u_m, u_1\rangle & \langle v_1, u_1\rangle & \langle v_2, u_1\rangle & \dots & \langle v_n, u_1\rangle \\
\langle u_1, u_2\rangle & \mathbf{1} & \dots & \langle u_m, u_2\rangle & \langle v_1, u_2\rangle & \langle v_2, u_2\rangle & \dots & \langle v_n, u_2\rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\langle u_1, u_m\rangle & \langle u_2, u_m\rangle & \dots & \mathbf{1} & \langle v_1, u_m\rangle & \langle v_2, u_m\rangle & \dots & \langle v_n, u_m\rangle \\
\langle u_1, v_1\rangle & \langle u_2, v_1\rangle & \dots & \langle u_m, v_1\rangle & \mathbf{1} & \langle v_2, v_1\rangle & \dots & \langle v_n, v_1\rangle \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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\end{pmatrix}$$

The matrices $\Gamma_V(u,v)$ and the Gram matrix I

Let $(V, \langle\cdot,\cdot\rangle)$ be an arbitrary inner product space over $\mathbb{F}$ and $m, n \in \mathbb{N}$. Let $u_1, u_2, \ldots, u_m \in V$ and $v_1, v_2, \ldots, v_n \in V$. We put

$$\Gamma_V(u,v)_{ij} := \langle v_j, u_i\rangle \qquad ((i,j) \in [m] \times [n]),$$

where $u := (u_1, \ldots, u_m) \in V^m$ and $v := (v_1, \ldots, v_n) \in V^n$.

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If $m = k = n$ and $u = w = v \in V^k$ then $\Gamma_V(w,w)$ is called Gram matrix of the vectors $w_1, w_2, \ldots, w_k \in V$.

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Hence, our approach actually is the following

The matrices $\Gamma_V(u, v)$ and the Gram matrix II

Observation

Let $m, n \in \mathbb{N}$ and $(V, \langle\cdot,\cdot\rangle)$ be an arbitrary inner product space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in V$. Put $u \oplus v := (u_1, \ldots, u_m, v_1, \ldots, v_n)$, where $u := (u_1, \ldots u_m)$ and $v := (v_1, \ldots v_n)$. Then

$$\Gamma_V(u \oplus v, u \oplus v) \overset{\checkmark}{=} \begin{pmatrix} \Gamma_V(u, u) & \Gamma_V(u, v) \\ \Gamma_V(u, v)^* & \Gamma_V(v, v) \end{pmatrix} .$$

A recollection of a few definitions I

Let $n \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then $PSD(n; \mathbb{F})$ denotes the set of all Hermitian positive semidefinite $n \times n$-matrices with entries in $\mathbb{F}$. Recall that a Hermitian matrix $A \in \mathbb{M}(n \times n; \mathbb{F})$ is called positive semidefinite if $\langle Ax, x \rangle_{\mathbb{F}^n} = x^* A x \geq 0$ for all $x \in \mathbb{F}^n$.

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Moreover, we consider the set

$$C(n; \mathbb{F}) := \{S \in PSD(n; \mathbb{F}) \text{ such that } S_{ii} = 1 \text{ for all } i \in [n]\}.$$

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An element of $C(n;\mathbb{F})$ is called correlation matrix. $C(n;\mathbb{R})$ is also known as "n-elliptope" in convex algebraic geometry.

A recollection of a few definitions II

Let $n \in \mathbb{N}$. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\xi := (\xi_1, \xi_2, \ldots, \xi_n)^\top : \Omega \longrightarrow \mathbb{R}^n$ be a random vector. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $C \in PSD(n; \mathbb{R})$.

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$$\langle a, \xi \rangle = \sum_{i=1}^{n} a_i \xi_i = \langle a, \mu \rangle + \sqrt{\langle a, Ca \rangle}\, \eta_a$$

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Note that we don't require here that C is invertible!

A recollection of a few definitions III

Following Feller, the real $n \times n$-matrix $\mathbb{V}(\xi) := \mathbb{E}[\xi\,\xi^\top]$ defined as

$$\mathbb{V}(\xi)_{ij} := \mathbb{E}[\xi\,\xi^\top]_{ij} := \mathbb{E}[\xi_i\xi_j] - \mathbb{E}[\xi_i]\mathbb{E}[\xi_j] \overset{(!)}{=} C_{ij} \qquad (i,j \in [n])$$

is known as the *variance matrix of the Gaussian random vector* ξ.

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is known as the *variance matrix of the Gaussian random vector* ξ.

However, if the variance matrix C is invertible then the density function of $\xi \sim N_n(\mu, C)$ exists and is given by

$$\varphi_{\mu,C}(x) := \frac{1}{(2\pi)^{n/2} \sqrt{\det(C)}} \exp\left(-\frac{1}{2}\langle x - \mu, C^{-1}(x - \mu)\rangle\right),$$

where $x \in \mathbb{R}^n$.

Complex Gaussian random vectors I

Definition

Let $\underline{Z} \equiv (X_1 + i\,Y_1, X_2 + i\,Y_2, \ldots, X_n + i\,Y_n)^\top$ be an n-dimensional complex random vector. Put $\xi := (X_1, \ldots, X_n)^\top, \eta := (Y_1, \ldots, Y_n)^\top$. Then $\underline{Z}$ is a complex Gaussian random vector if the real $2n$-dimensional random vector $(\xi^\top, \eta^\top)^\top$ is a Gaussian random vector.

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Although this definition looks rather innocently, it isn't ! It contains a lot of (hidden) structure which extends the real Gaussian case *by far*.

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Although this definition looks rather innocently, it isn't ! It contains a lot of (hidden) structure which extends the real Gaussian case *by far*.

Since $(\xi^\top, \eta^\top)^\top$ is a real Gaussian random vector the complex matrices $\Gamma := \mathbb{E}[\underline{Z}\underline{Z}^*]$ (variance matrix) and $C := \mathbb{E}[\underline{Z}\underline{Z}^\top]$ both are well-defined, such as $\mu := \mathbb{E}[\underline{Z}]$.

Complex Gaussian random vectors II

Hence, the distribution of $(\xi^\top, \eta^\top)^\top$ is fully specified by μ, Γ and C. We write $\underline{Z} \sim \mathbb{C}N_n(\mu, \Gamma, C)$ if $\underline{Z}$ is an n-dimensional complex Gaussian random vector.

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A very important special case is given when $\mu = 0$ and $C = 0$. In this case $\mathbb{C}N_n(0, \Gamma, 0)$ is shortened to $\mathbb{C}N_n(0, \Gamma)$. $\underline{Z} \sim \mathbb{C}N_n(0, \Gamma)$ is known as proper complex Gaussian random vector (with variance matrix Γ). The latter is a well-known definition in electrical engineering science.

Structure of $C(n;\mathbb{R})$

Proposition

Let $n \in \mathbb{N}$ *and* $\Sigma = (\sigma_{ij}) \in \mathbb{M}(n \times n; \mathbb{R})$*. TFAE:*

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Some geometry of correlation matrices

Observation

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k \in \mathbb{N}$. *Then the sets* $\{S : S = xx^* = \Gamma_{\mathbb{F}}(x, x)$ *for some* $x \in (\mathbb{S}^0)^k\}$ *and* $\{\Theta : \Theta \in C(k; \mathbb{F})$ *and rk*$(\Theta) = 1\}$ *coincide.*

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Proposition (K. R. Parthasarathy (2002))

Let $k \in \mathbb{N}$. *The* k*-elliptope* $C(k; \mathbb{R})$ *is a compact and convex subset of the* k^2*-dimensional vector space* $\mathbb{M}(k \times k; \mathbb{R})$. *Any* $k \times k$*-correlation matrix of rank* 1 *is an extreme point of the set* $C(k; \mathbb{R})$ *(yet not conversely).*

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In particular, the (finite) set of all real $k \times k$-correlation matrices of rank 1 is not convex.

Let $k \in \mathbb{N}$. Put

$$C_1(k;\mathbb{F}) := \{\Theta : \Theta \in C(k;\mathbb{F}) \text{ and } \mathrm{rk}(\Theta) = 1\}.$$

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Having “embedded” the $m \times n$-matrix $\Gamma_H(u, v)$ in a $(m+n) \times (m+n)$- correlation matrix, how could this gained information be used to reformulate Grothendieck’s inequality accordingly? To answer this question, let us also “embed” the $m \times n$-matrix A suitably!

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Definition

Let $m, n \in \mathbb{N}$ and $A \in \mathbb{M}(m \times n; \mathbb{F})$ arbitrary. Put

$$\widehat{A} := \frac{1}{2} \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

Let us call $\mathbb{M}((m+n) \times (m+n); \mathbb{F}) \ni \widehat{A}$ the canonical block injection of A.

A further reformulation of GT

Proposition

Let H be an arbitrary Hilbert space over $\mathbb{F}$. Let $m, n \in \mathbb{N}$ and $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$. Let $K > 0$. TFAE:

(i)

$$\max_{\|u_i\|=1=\|v_j\|} \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle u_i, v_j \rangle_H \Big| \leq K \max_{|p_i|=1=|q_j|} \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \Big| .$$

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Note that we don't know whether condition (ii) holds for all matrices in $\mathbb{M}((m+n) \times (m+n); \mathbb{F})$. If this were the case, GT would turn out to be a corollary of a more general statement!

GT versus NP-hard optimisation

Observation (Real Case: $\mathbb{F} = \mathbb{R}$)

On the left side of GT: a convex conic optimisation problem (since it is SDP) and hence of polynomial worst-case complexity (P)):

$$\max_{\Sigma \in C(m+n;\mathbb{R})} |\langle \widehat{A}, \Sigma \rangle|$$

On the right side: an NP-hard, non-convex combinatorial (Boolean) optimisation problem:

$$\max_{\substack{\Theta \in C(m+n;\mathbb{R}) \\ rk(\Theta)=1}} |\langle \widehat{A}, \Theta \rangle|$$

Thus, Grothendieck's constant $K_G^{\mathbb{R}}$ is precisely the "integrality gap"; i. e., the maximum ratio between the solution quality of the NP-hard Boolean optimisation on the right side of GT and of its SDP relaxation on the left side!

1. *An important information for readers*
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Schur product and the matrix $f[A]$ I

Definition

Let $\emptyset \neq U \subseteq \mathbb{F}$ and $f : U \longrightarrow \mathbb{F}$ be a function. Let $A = (a_{ij}) \in \mathbb{M}(m \times n; \mathbb{F})$ such that $a_{ij} \in U$ for all $(i, j) \in [m] \times [n]$. Define $f[A] \in \mathbb{M}(m \times n; \mathbb{F})$ - entrywise - as $f[A]_{ij} := f(a_{ij})$ for all $(i, j) \in [m] \times [n]$.

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Guiding Example

The Schur product (aka Hadamard product)

$$(a_{ij}) * (b_{ij}) := (a_{ij} b_{ij})$$

of matrices (a_{ij}) *and* (b_{ij}) *leads to matrices* $A^{*\,k} = f[A]$, *where* $f(x) := x^k$ $(k \in \mathbb{N})$.

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The notation "$f[A]$" is used to highlight the difference between the matrix $f(A)$ originating from the spectral representation of A (for normal matrices A) and the matrix $f[A]$, defined as above !

Schur product and the matrix $f[A]$ II

Theorem

For any $n \in \mathbb{N}$ the set $\mathbb{M}(n \times n; \mathbb{F})$ with the usual addition and the Schur multiplication $*$ is a commutative Banach algebra under the operator norm.

Schur product and the matrix $f[A]$ II

Theorem

For any $n \in \mathbb{N}$ the set $\mathbb{M}(n \times n; \mathbb{F})$ with the usual addition and the Schur multiplication $*$ is a commutative Banach algebra under the operator norm. Moreover,

$$PSD(n; \mathbb{F}) * PSD(n; \mathbb{F}) \subseteq PSD(n; \mathbb{F}) \, .$$

In particular, $C(n; \mathbb{F}) * C(n; \mathbb{F}) \subseteq C(n; \mathbb{F})$.

Grothendieck's identity I

Theorem (Grothendieck's identity - T. S. Stieltjes (1889), W. F. Sheppard (1898), A. Grothendieck (1953))

Let $-1 \leq \rho \leq 1$ *and* $(X, Y)^\top \sim N_2(0, \Sigma_2(\rho))$, *where*

$$\Sigma_2(\rho) := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Consider the function sign $: \mathbb{R} \longrightarrow \{-1, 1\}$, *defined as sign*$(x) := 1$ *if* $x \geq 0$ *and sign*$(x) := -1$ *else. Then*

$$\mathbb{E}[\mathit{sign}(X)\mathit{sign}(Y)] = \frac{2}{\pi}\arcsin(\rho) = \frac{2}{\pi}\arcsin\left(\mathbb{E}[XY]\right).$$

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The following implication of Grothendieck's identity can be read often in related papers:

Grothendieck's identity II

Corollary

Let $n \in \mathbb{N}$*,* $u, v \in \mathbb{S}^{n-1}$ *and* $\xi \sim N_n(0, I_n)$*. Then*

$$\mathbb{E}[\mathit{sign}(u^\top \xi)\mathit{sign}(v^\top \xi)] = \frac{2}{\pi} \arcsin\left(u^\top v\right) .$$

Gauss is lurking!

Observation

Let $\rho \in [-1, 1]$. *Then*

$$\begin{aligned} \frac{2}{\pi}\arcsin(\rho) &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{\rho^{2n+1}}{2n+1} \\ &\overset{!}{=} \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,n!}\frac{\rho^{2n+1}}{2n+1} \overset{\checkmark}{=} \frac{2}{\pi}\,\rho\cdot{}_2F_1\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};\rho^2\right), \end{aligned}$$

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where

$${}_2F_1(a,b,c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\,n!}z^n,$$

$z \in \overline{\mathbb{D}}$ *and* $a,b,c \in \mathbb{F}$, *satisfying* $-c \notin \mathbb{N}_0$ *and* $\Re(c-a-b) > 0$, *denotes the famous, classical Gauss hypergeometric function.*

Grothendieck's identity in matrix form I

Corollary

Let $k \in \mathbb{N}$ *and* $\Sigma \in C(k;\mathbb{R})$ *be an arbitrarily given* $(k \times k)$*-correlation matrix. Then also* $\frac{2}{\pi}\arcsin[\Sigma] \in C(k;\mathbb{R})$ *. There exists a Gaussian random vector* $\xi \sim N_k(0,\Sigma)$ *such that*

$$\frac{1}{\arcsin(1)}\arcsin[\Sigma] = \frac{2}{\pi}\arcsin[\Sigma] = \mathbb{E}\left[\Theta(\xi)\right],$$

where

$$\Theta(\xi(\omega))_{ij} := \textit{sign}(\xi_i(\omega))\textit{sign}(\xi_j(\omega))$$

for all $\omega \in \Omega$*, and for all* $i,j \in [k]$*.* $\Theta(\xi(\omega))$ *is a correlation matrix of rank* 1 *for all* $\omega \in \Omega$*.*

Grothendieck's identity in matrix form II

Corollary

Let $k = m + n$, *where* $m, n \in \mathbb{N}$. *Let* $\Sigma \in C(k; \mathbb{R})$ *be an arbitrarily given* $(k \times k)$*-correlation matrix. Then*

$$\frac{2}{\pi} |\langle \widehat{A}, \arcsin[\Sigma] \rangle| = |\langle \widehat{A}, \mathbb{E}[\Theta(\xi)] \rangle| \leq \mathbb{E}[|\langle \widehat{A}, \Theta(\xi) \rangle|] \leq \max_{\substack{\Theta \in C(k;\mathbb{R}) \\ rank(\Theta)=1}} |\langle \widehat{A}, \Theta \rangle| \, .$$

for all matrices $A \in M(m \times n; \mathbb{R})$.

The complex case: Haagerup's identity revisited I

Lemma (U. V. Haagerup (1987))

Let $z \in \mathbb{D}$ *and* $(Z, W)^\top \sim \mathbb{C}N_2(0, \Sigma_2(z))$. *Then*

$$\mathbb{E}[\textit{sign}(Z)\textit{sign}(\overline{W})] = z \int_0^{\pi/2} \frac{\cos^2(u)}{\sqrt{1 - |z|^2 \sin^2(u)}}\, du$$

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Proposition

Let $z \in \mathbb{D}$ *and* $(Z, W)^\top \sim \mathbb{C}N_2(0, \Sigma_2(z))$ *and* $f(x, y) := \langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \rangle$ *for all* $(x, y) \in \mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}$. *Then*

$$\mathbb{E}[\mathit{sign}(Z)\mathit{sign}(\overline{W})] = \mathit{sign}(z) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y) \, \varphi_{0, \Sigma_4(|z|)}(x, y) \, d^2x \, d^2y \, .$$

The complex case: Haagerup's identity revisited II

Theorem

Let $z \in \overline{\mathbb{D}}$ *and* $(Z, W)^\top \sim \mathbb{C}N_2(0, \Sigma_2(z))$. *Then*

$$\mathbb{E}[\mathit{sign}(Z)\mathit{sign}(\overline{W})] = \frac{\pi}{4}\, z \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2; |z|^2\right).$$

The complex case: Haagerup's identity revisited II

Theorem

Let $z \in \overline{\mathbb{D}}$ *and* $(Z, W)^\top \sim \mathbb{C}N_2(0, \Sigma_2(z))$. *Then*

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Hence,

A unification of the real and the complex case

Proposition

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k = m + n$*, where* $m, n \in \mathbb{N}$*. Let* H *be an arbitrary Hilbert space. Then*

$$\max_{u \in S_H^m, v \in S_H^n} \left| tr\left(A\, f_{\mathbb{F}}[\Gamma_H(u,v)^*]\right) \right| = |\langle \widehat{A}, f_{\mathbb{F}}[\Sigma^*] \rangle| \leq c_{\mathbb{F}} \max_{\substack{\Theta \in C(k;\mathbb{F}) \\ rank(\Theta) = 1}} |\langle \widehat{A}, \Theta \rangle|\,,$$

for all matrices $A \in M(m \times n; \mathbb{F})$*, where*

$$f_{\mathbb{R}}(\rho) := \rho \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \rho^2\right) \ (\rho \in [-1,1]) \text{ and } c_{\mathbb{R}} := \frac{\pi}{2} \overset{(!)}{=} f_{\mathbb{R}}(1)$$

and

$$f_{\mathbb{C}}(z) := |z| \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2; |z|^2\right) \ (z \in \overline{\mathbb{D}}) \text{ and } c_{\mathbb{C}} := \frac{4}{\pi} \overset{(!)}{=} f_{\mathbb{C}}(1)$$

CCP functions I

A Natural Question

Let $k = m + n$, where $m, n \in \mathbb{N}$. How do we have to change our approach to obtain the following:

$$|\langle \widehat{A}, \Sigma \rangle| \leq \widetilde{c_{\mathbb{F}}} \max_{\substack{\Theta \in C(k;\mathbb{F}) \\ rank(\Theta)=1}} |\langle \widehat{A}, \Theta \rangle| \,,$$

for all $\Sigma \in C(k; \mathbb{F})$ and all $A \in M(k; \mathbb{F})$? $\widetilde{c_{\mathbb{F}}} = ?$

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Key Idea

Inversion of the power series $f_{\mathbb{F}}$?

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Inversion of the power series $f_{\mathbb{F}}$? Unfortunately, the mapping

$$C(k; \mathbb{F}) \longrightarrow C(k; \mathbb{F}), \Sigma \mapsto f_{\mathbb{F}}[\Sigma]$$

in general is not onto (else $1,676 < K_G^{\mathbb{R}} \leq \frac{\pi}{2} \approx 1,571$ - a contradiction)!

CCP functions I

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in general is not onto (*else* $1,676 < K_G^{\mathbb{R}} \leq \frac{\pi}{2} \approx 1,571$ *- a contradiction*)*! However, inversion seemingly is not a bad idea...*

CCP functions II

Following the important concept of completely positive maps (which however by definition are assumed to be linear) we introduce the following

Definition

Let $n \in \mathbb{N}$ and $g : [-1, 1] \longrightarrow \mathbb{R}$ be an arbitrary function.

(i) g is n-correlation-preserving (short: n-CP) if $g[\Sigma]$ is an $(n \times n)$-correlation matrix for all $(n \times n)$-correlation matrices Σ.

(ii) g is called completely correlation-preserving (short: CCP) if g is n-correlation-preserving for all $n \in \mathbb{N}$.

CCP functions III

Theorem (Schoenberg (1942), Rudin (1959), Berg, Christensen, Ressel (1984), Guillot, Khare and Rajaratnam (2016))

Let $g : [-1, 1] \longrightarrow \mathbb{R}$ be an arbitrary continuous function. TFAE:

(i) g *is CCP.*

(ii) $g(1) = 1$ *and* $g[A] \in PSD(n; \mathbb{R})$ *for all* $A \in PSD(n; [-1, 1])$ *and* $n \in \mathbb{N}$.

(iii) g *admits a power series representation* $g(t) = \sum_{k=0}^{\infty} b_k t^k$ *on* $[-1, 1]$ *for some sequence* (b_n)*, consisting of non-negative numbers only and* $\sum_{k=0}^{\infty} b_k = 1$.

A correlation matrix preserving cross transformation

Lemma

Let $f, g : [-1, 1] \longrightarrow \mathbb{R}$ be two functions satisfying certain ”correlation preserving conditions“. Let $m, n \in \mathbb{N}$ and

$$\Sigma := \begin{pmatrix} C & S \\ S^\top & D \end{pmatrix}$$

be an arbitrary $(m + n) \times (m + n)$-correlation matrix. Assume that $f(c^*) = 1$ for some $0 < c^* \leq 1$.

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be an arbitrary $(m+n) \times (m+n)$-correlation matrix. Assume that $f(c^*) = 1$ for some $0 < c^* \leq 1$. Then also the matrix

$$\begin{pmatrix} f[c^* C] & g[c^* S] \\ g[c^* S^\top] & f[c^* D] \end{pmatrix}$$

is a $(m+n) \times (m+n)$-correlation matrix.

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Krivine revisited I

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$$\widetilde{\Sigma} := \begin{pmatrix} \sinh[c^* C] & \sin[c^* S] \\ \sin[c^* S^\top] & \sinh[c^* D] \end{pmatrix} \overset{(!)}{=}$$

Krivine revisited I

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is a $(m+n) \times (m+n)$-correlation matrix, where g denotes the CCP function $g(\rho) := \frac{2}{\pi} \arcsin(\rho) = \frac{f_{\mathbb{R}}(\rho)}{f_{\mathbb{R}}(1)}$, $-1 \leq \rho \leq 1$.

Krivine revisited II

Example ctd.

Consequently, since g is CCP,

$$\begin{pmatrix} g \circ \sinh[\frac{\pi}{2} r^* C] & r^* S \\ r^* S^\top & g \circ \sinh[\frac{\pi}{2} r^* D] \end{pmatrix} = g[\widetilde{\Sigma}] = \frac{2}{\pi} f_{\mathbb{R}}[\widetilde{\Sigma}]$$

is a $(m+n) \times (m+n)$-correlation matrix, too.

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is a $(m+n) \times (m+n)$-correlation matrix, too. Hence,

$$\begin{aligned} \Big| |tr(A\, S^\top)| &= \frac{1}{r^*} |tr(A\, (r^*\, S)^\top)| \\ &= \frac{1}{r^*} |\langle \widehat{A}, \frac{2}{\pi} f_{\mathbb{R}}[\widetilde{\Sigma}] \rangle| \le \frac{1}{r^*} \max_{\Theta \in C_1(k;\mathbb{R})} |\langle \widehat{A}, \Theta \rangle| \,. \end{aligned}$$

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Consequently,

$$K_G^{\mathbb{R}} \leq \frac{1}{r^*} = \frac{\pi}{2\ln(1+\sqrt{2})} = \frac{\sin^{-1}(1)}{\sinh^{-1}(1)} \,.$$

A first generalisation of Krivine's approach - sketch I

In the following we consider the case $\mathbb{F} = \mathbb{R}$.

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$$\mathbb{E}[b(X)\, b(Y)] \overset{!}{=} \rho \sum_{n=0}^{\infty} (a_{2n+1}(b))^2\, (\rho^2)^n =: g_b(\rho)$$

for some sequence $(a_n(b))_{n \in \mathbb{N}_0} \in S_{l_2}$.

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$$\mathbb{E}[b(X)\, b(Y)] \stackrel{!}{=} \rho \sum_{n=0}^{\infty} (a_{2n+1}(b))^2 \, (\rho^2)^n =: g_b(\rho)$$

for some sequence $(a_n(b))_{n \in \mathbb{N}_0} \in S_{l_2}$. Hence, g_b is a CCP function (Schoenberg!).

(ii) Assume that $a_1(b) \neq 0$ (already implying that g_b has an inverse function g_b^{-1}, defined around 0).

A first generalisation of Krivine's approach - sketch II

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Observation

Given all of the assumptions (i) - (vi) above, we have:

$$K_G^{\mathbb{R}} \leq \frac{1}{r^*}\,.$$

A phrase of G. H. Hardy

"... at present I will say only that if a chess problem is, in the crude sense, 'useless', then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects..."

– *A Mathematician's Apology* (1940)

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Thank you for your attention!

Are there any questions, comments or remarks?