The stochastic logarithm of semimartingales and market prices of risk

Frank Oertel
http://euclid.ucc.ie/pages/staff/oertel

AMaMeF Workshop on Stochastic Analysis and Computational Finance

Institute for Mathematical Sciences
Imperial College London

12th November 2005

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The Standard Black-Merton-Scholes Model Revisited

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a given probability space, $T < \infty$ a finite time horizon and $W = (W_t)_{0 \le t \le T}$ be a standard Brownian motion in $\mathbb{R}$ (SBM). Information is modelled as the augmented natural filtration $\mathbb{F}^W = \{\mathscr{F}_t^W \mid 0 \le t \le T\}$ of the SBM W, where $\mathscr{F}_T^W = \mathscr{F}$.

Consider two tradable assets:

- A riskless bond with (non-stochastic) price process $B = (B_t)_{0 \le t \le T}$;

- A stock with stochastic price process $S = (S_t)_{0 \le t \le T}$.

B is modelled as $B_t := e^{rt}$, where $r > 0$ (risk-free rate of interest), and S follows a geometric Brownian motion:

$$dS_t = S_t \, dR_t,$$

where $R_t := \mu t + \sigma W_t$ denotes the cumulated return process ($\mu \geq 0$ and $\sigma > 0$). Thus:

$$\begin{aligned} S_t &\overset{\text{(Itô)}}{=} S_0 \exp\Big(\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t\Big) \\ &= S_0 \exp\Big(R_t - \frac{1}{2}[R,R]_t\Big) \\ &= S_0 \, \mathscr{E}(R)_t \, . \qquad \Big[\longleftarrow \text{stochastic exponential}\Big] \end{aligned}$$

Let $f \in L^0(\Omega, \mathscr{F}, \mathbb{P})$ be an arbitrary contingent claim such as a European call option, a credit swap or a reverse convertible bond.

Problem: Does there exists a consistent price system for f, and is it possible to eliminate (large parts of) the intrinsic risk of f by replicating f through a hedging portfolio?

Well-known Idea: We move to a 'risk-neutral universe', where no 'money-machines' exist and the stock has no 'drift'!

Theorem 1 *Assume the standard BMS-model and let $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T^W$. TFAE:*

(i) *The discounted price process $S^B := \frac{S}{B}$ is a $\mathbb{Q}$-martingale;*

(ii) $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, *where* $Z_T = \mathscr{E}\Big(- \dfrac{\mu - r}{\sigma} W \Big)_T$.

$\dfrac{\mu - r}{\sigma}$ is known as the Girsanov kernel resp. 'market price of risk' or 'Sharpe ratio'.

Since $\mathbb{Q} \sim \mathbb{P}$, $\mathbb{Q}$ is an equivalent martingale measure (EMM).

Hence again, in the standard BMS-model there exists a unique EMM, consequently a unique risk-neutral price. The BMS-model is complete.

Imperfections of the BMS-model:

- Log returns do not behave according to a Normal distribution;

- Volatilities change stochastically over time and are clustered.

Therefore, we turn our attention to...

Semimartingales with Lévy Processes in View

From now on, we assume that a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is given which in addition is endowed with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ satisfying the usual conditions (such as $\mathbb{F}^W$).

Definition 1 (Semimartingale) *A process $X = (X_t)$ in $\mathbb{R}^d$ whose path is right continuous and has left limits ('càdlàg' resp. 'RCLL') is called a semimartingale (w.r.t. $\mathbb{P}$ and $\mathbb{F}$) if there exist an adapted local martingale M and an adapted process A of finite variation such that*

$$X = M + A\,.$$

If in addition the process A is predictable and $A_0 = 0$, the decomposition of X even is unique. In this case, X is called a special semimartingale and $\widetilde{X} := A$ the compensator of X.

Examples: Standard Brownian motion, geometric Brownian motion, the Poisson process, and the compound Poisson process are semimartingales.

Fractional Brownian motion (if $H \neq \frac{1}{2}$) and processes of type $|W|^{\alpha}$, where $0 < \alpha < 1$ and W is a standard Brownian motion, do not belong to the class of semimartingales.

Stability properties: The class $\mathscr{S}$ of semimartingales has a vector space structure. $\mathscr{S}$ can be equipped with a topology such that it even turns into a Fréchet space. Due to Itô's formula, $\mathscr{S}$ is an algebra and stable under C^2-functions on open sets. Moreover, $\mathscr{S}$ is stable under convex functions. In particular, it is a lattice (e. g., $X \wedge Y = \frac{1}{2}(X + Y - |X - Y|)$).

Definition 2 (Lévy Process) *A process* $X = (X_t)_{t \geq 0}$ *in* $\mathbb{R}^d$ *is a* $\mathbb{F}$-*Lévy process if*

(i) X *is càdlàg;*

(ii) $X_0 = 0$ *a.s.;*

(iii) X *is adapted and has independent increments w.r.t.* $\mathbb{F}$ (*independent from the past*);

(iv) *If* $0 \leq s \leq t$, *then* $X_t - X_s \stackrel{d}{=} X_{t-s}$ (*stationary increments*).

In the following, we only consider the case $d = 1$ (to simplify notation).

Examples: Brownian motion, the Poisson process, Cauchy and α-stable processes belong to the class of Lévy processes.

Processes of type $(at + \sigma W_t + Y_t)_{t \geq 0}$, where $a, \sigma \in \mathbb{R}$, Y is a compound Poisson process and W is a SBM (in $\mathbb{R}$), are Lévy processes as well.

Further examples: Variance Gamma (VG), Normal Inverse Gaussian (NIG), CGMY, the Hyperbolic model, and the Meixner process.

Let $X = (X_t)$ be an arbitrary Lévy process in $\mathbb{R}$. Let $u \in \mathbb{R}$ and $t \geq 0$. Put

$$\varphi_{X_t}(u) = \widehat{\mathbb{P}_{X_t}}(u) := \mathbb{E}[e^{iuX_t}] = \int_{\mathbb{R}} e^{iux}\, \mathbb{P}_{X_t}(dx)\,.$$

Theorem 2 (Lévy-Khintchine Formula) *Let X be a Lévy process. Then there exists a Radon measure ν on $\mathscr{B}(\mathbb{R})$ such that $\nu(\{0\}) = 0$, $\int\limits_{\mathbb{R}} x^2 \wedge 1 \, \nu(dx) < \infty$, and parameters $\alpha \in \mathbb{R}$, $\sigma \geq 0$ such that*

$$\varphi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)} = \varphi^t_{X_1}(u), \qquad (*)$$

where

$$\psi(u) := \frac{\sigma^2 u^2}{2} - i\alpha u + \int\limits_{\mathbb{R}} \left(1 - e^{iux} + iux \mathbb{1}_{(-1,1)}(x)\right) \nu(dx).$$

Conversely, given any such triplet (α, σ^2, ν), there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a corresponding Lévy process X with characteristic function $(*)$ - unique in distribution.

The (non-random) triple (α, σ^2, ν) is called the local characteristics of X. ν is known as the Lévy measure. Note that the local characteristics depends on X. Consequently, we also make use of the notation $(\alpha^X, (\sigma^X)^2, \nu^X)$.

Remark and Definition 1 *Let X be an arbitrary adapted and càdlàg process.*

$$\Delta X_s := X_s - X_{s-} = X_s - \lim_{u \uparrow s} X_u$$

denotes the jump of X at time $s \geq 0$, where we put $X_{0-} := X_0$. All paths of X are continuous except for the jumps. Moreover, X_- is left-continuous, locally bounded and predictable, hence adapted. ΔX denotes the corresponding jump process. If X is continuous, then of course, $\Delta X = 0$.

For every $\omega \in \Omega$, the random set

$$M(\omega) \ := \ \left\{ (s, \Delta X_s(\omega)) \middle| s > 0, \Delta X_s(\omega) \neq 0 \right\}$$

is at most countable.

The jump measure of X is defined as the mapping

$$j_X : \underbrace{\Omega}_{\text{samples}} \times \mathscr{B}(\underbrace{\mathbb{R}^+}_{\text{time}} \times \underbrace{\mathbb{R}}_{\text{range of } X}) \longrightarrow \mathbb{N}_0 \cup \{\infty\},$$

$$j_X(\omega, B) := \text{card}(M(\omega) \cap B),$$

where $\omega \in \Omega$ and B is an arbitrary Borel set in $\mathbb{R}^+ \times \mathbb{R}$. j_X counts, ω-by-ω, the number of all $s > 0$ such that $\Delta X_s(\omega) \neq 0$ and $(s, \Delta X_s(\omega)) \in B$.

In particular, if X is a Lévy process, the set $B := [0, t] \times \Lambda$ (where t is fixed and Λ is a Borel set in $\mathbb{R}$) induces the random measure

$$N_t^\Lambda(\omega) := N_t(\omega, \Lambda) := j_X(\omega, [0, t] \times \Lambda)$$

which - in the Lévy case (!) - even leads to a Poisson process $N^\Lambda = (N_t^\Lambda)_{t \geq 0}$ of intensity $\nu(\Lambda)$:

$$\mathbb{E}[N_t^\Lambda] = \mathbb{E}[N_t(\cdot, \Lambda)] = t\nu(\Lambda) \text{ for all } t \geq 0.$$

Theorem 3 (Lévy Decomposition) *Let X be a Lévy process with local characteristics (α, σ^2, ν). Then*

$$X = A + M^c + M^d$$

is the sum of three independent Lévy processes, where

(i) $A_t := \alpha t + \sum_{i=1}^{N_t^{\Lambda}} Y_i$ *is a finite variation compound Poisson process with drift having only jumps of size at least 1 (where $\Lambda := \mathbb{R} \setminus (-1, 1)$ and $Y_i := \Delta X_{T_i^{\Lambda}}$),*

(ii) $M_t^c := \sigma W_t \overset{d}{=} W_{\sigma^2 t}$ *is a Brownian motion,*

(iii) $M_t^d := \int_{(-1,1)} x(N_t(\cdot, dx) - t\nu(dx))$ *is a pure-jump martingale having only jumps of size less than 1.*

Corrolary 1 *A Lévy process is a semimartingale.*

The Stochastic Logarithm: The Germ of Financial Mathematics?

Theorem 4 (Stochastic Exponential) *Let X be an $\mathbb{R}$-valued semimartingale such that $X_0 = 0$. Put*

$$V_t := \prod_{0<s\leq t} (1 + \Delta X_s) \cdot e^{-\Delta X_s} .$$

Then for almost all $\omega \in \Omega$ the infinite product is absolutely convergent, and $V = (V_t)$ is an adapted purely discontinuous process which is of finite variation. Put

$$Z_t := \exp\left(X_t - \frac{1}{2}[X,X]_t^c\right) \cdot V_t .$$

Then $Z_0 = 1$, and $Z = (Z_t)$ is the unique semimartingale which is a solution of the following stochastic integral equation

$$Z = 1 + \int Z_- \, dX .$$

Moreover, $Z = Z_-(1 + \Delta X)$.

The semimartingale Z is called the stochastic exponential of the semimartingale X and is denoted by $\mathscr{E}(X)$. If X is a local martingale, so is $\mathscr{E}(X)$.

Examples: If W is a standard Brownian motion, $\mathscr{E}(W)_t = \exp(W_t - \frac{1}{2}t)$, and if N is a Poisson process, then $\mathscr{E}(N)_t = 2^{N_t}$.

Let X be a Lévy process with local characteristics (α, σ^2, ν). Assume that the Laplace transform $\mathbb{E}[e^{-zX_1}] < \infty$ for all z in some $B_r(0)$. Then $X_t = \alpha t + \sigma W_t + M_t$, where $\alpha := \mathbb{E}[X_1] < \infty$ and $M_t := \int\limits_{\mathbb{R}} x(N_t(\cdot, dx) - t\nu(dx))$. Let $dS_t = S_{t-}(\mu_t dt + \rho_t dX_t)$, where μ and ρ are continuous and deterministic functions. Then

$$
\begin{aligned}
S_t &= S_0\,\mathscr{E}\Big(\int_0^{\bullet} \mu_s\,ds + \int \rho\,dX\Big)_t \\
&= S_0 \exp\Big(\int_0^t \sigma\rho_s\,dW_s + \int_0^t \rho_s\,dM_s \\
&+ \int_0^t \Big(\alpha\rho_s + \mu_s - \frac{\sigma^2{\rho_s}^2}{2}\Big)\,ds\Big) \\
&\times \prod_{0<s\le t} (1 + \rho_s \Delta M_s) \cdot \exp(-\rho_s \Delta M_s).
\end{aligned}
$$

The mapping $X \mapsto \mathscr{E}(X) = Z$ can be inverted if almost all paths of Z and Z_- do not go through 0 (cf. Jacod (1979), Foldes (1990), Choulli et al (1998), Kallsen/Shiryaev (2001), Jacod/Shiryaev (2nd ed. 2002)).

Theorem 5 (Stochastic Logarithm) *Let*

$$\mathscr{S}_0^{\sim} := \Big\{ X \in \mathscr{S} \,\Big|\, \{\, \Delta X = -1\} \text{ is evanescent}, X_0 = 0 \Big\}$$

and

$$\mathscr{S}_1^{*} := \Big\{ Z \in \mathscr{S} \,\Big|\, \{ZZ_- = 0\} \text{ is evanescent, } Z_0 = 1 \Big\}.$$

Then the mapping

$$\begin{array}{rcl} \mathscr{E} : \mathscr{S}_0^{\sim} & \xrightarrow{\simeq} & \mathscr{S}_1^{*} \\ X & \mapsto & \mathscr{E}(X) \end{array}$$

is bijective, and its inverse is given by

$$\begin{array}{rcl} \mathscr{L} : \mathscr{S}_1^{*} & \xrightarrow{\simeq} & \mathscr{S}_0^{\sim} \\ Z & \mapsto & \displaystyle\int \frac{1}{Z_-} dZ - 1\,. \end{array}$$

In analogy to real analysis, $\mathscr{L}(Z) := \int \frac{1}{Z_-}\,dZ - 1$ is called the stochastic logarithm of Z. Note that $\Delta\mathscr{L}(Z) = \frac{1}{Z_-}\Delta Z$.

Corrolary 2 *Let $Z \in \mathscr{S}_1^*$. TFAE:*

(i) *Z is a local martingale;*

(ii) *There exists a unique local martingale $M \in \mathscr{S}_0^\sim$ such that $Z = \mathscr{E}(M)$ (outside some evanescent set);*

(iii) *$\mathscr{L}(Z)$ is a local martingale.*

Proposition 1 *Let $U, V \in \mathscr{S}_1^*$. Then $UV \in \mathscr{S}_1^*$ and*

$$\mathscr{L}(UV) = \mathscr{L}(U) + \mathscr{L}(V) + \left[\mathscr{L}(U), \mathscr{L}(V)\right].$$

Moreover, $\frac{1}{V} \in \mathscr{S}_1^*$ and

$$\mathscr{L}\Big(\frac{1}{V}\Big) = 1 - \mathscr{L}(V) - \left[V, \frac{1}{V}\right].$$

In particular, $(\mathscr{S}_1^*, \cdot)$ is a commutative group.

Theorem 6 (Modelling of Asset Prices) *Let $S \in \mathscr{S}$ such that $S_0 \neq 0$ a.s. and $\frac{S}{S_0} \in \mathscr{S}_1^*$ (asset price process). Then there exists a unique semimartingale $R \in \mathscr{S}_0^\sim$ (the cumulated return process) such that $S = S_0\,\mathscr{E}(R)$, namely $R = \mathscr{L}\big(\frac{S}{S_0}\big)$.*

Let $N \in \mathscr{S}_1^*$ (numeraire – e. g., a price process of Euros) and put $S^N := \frac{S}{N}$ (e. g., the Euro price process of S). Then $\frac{S^N}{S_0} \in \mathscr{S}_1^*$, and

$$S^N = S_0 \mathscr{E}(Y_N),$$

where

$$Y_N := \mathscr{L}(\frac{1}{N}) + R + \Big[\int N_- dR, \frac{1}{N}\Big] = \mathscr{L}\Big(\frac{S^N}{S_0}\Big) \in \mathscr{S}_0^\sim .$$

Lemma 1 *Let $X \in \mathscr{S}$ and $Z \in \mathscr{S}_1^*$. Then $\frac{1}{Z} \in \mathscr{S}_1^*$, and*

$$\begin{aligned}\int \frac{1}{Z} d[Z,X] &= X_0 + \Big[\mathscr{L}(Z), X\Big]^c + \sum_{0<s\leq \bullet} \frac{\Delta Z_s}{Z_s} \Delta X_s \\ &= X_0 - \Big[\mathscr{L}\Big(\frac{1}{Z}\Big), X\Big].\end{aligned}$$

Proof. WLOG, let $X_0 = 0$. Then

$$\begin{aligned}
\int \frac{1}{Z}\, d[Z,X] &= \int \frac{1}{Z_-}\, d[Z,X] + \int \Delta\Big(\frac{1}{Z}\Big)\, d[Z,X] \\
&= \Big[\mathscr{L}(Z), X\Big] + \sum_{0<s\le\bullet} \Delta\Big(\frac{1}{Z}\Big)_s \Delta[Z,X]_s \\
&= \Big[\mathscr{L}(Z), X\Big] + \sum_{0<s\le\bullet} \Delta\Big(\frac{1}{Z}\Big)_s \Delta Z_s \Delta X_s \,.
\end{aligned}$$

Since $\Delta\mathscr{L}(Z) = \frac{\Delta Z}{Z_-}$, we have

$$\Big[\mathscr{L}(Z), X\Big] = \Big[\mathscr{L}(Z), X\Big]^c + \sum_{0<s\le\bullet} \frac{\Delta Z_s}{Z_{s-}} \Delta X_s \,,$$

and the first equation follows.

On the other hand, due to Proposition 1,

$$\begin{aligned}
\Big[\mathscr{L}\Big(\frac{1}{Z}\Big), X\Big] &= \Big[1 - \mathscr{L}(Z) - \Big[Z, \frac{1}{Z}\Big], X\Big] \\
&= -[\mathscr{L}(Z), X] - \Big[\Big[Z, \frac{1}{Z}\Big], X\Big] \\
&\overset{!}{=} -[\mathscr{L}(Z), X] - \sum_{0<s\le\bullet} \Delta Z_s \Delta\Big(\frac{1}{Z}\Big)_s \Delta X_s \\
&= -\int \frac{1}{Z}\, d[Z,X] \,.
\end{aligned}$$

□

Theorem 7 (Log-Version of Girsanov) *Let $\mathbb{Q}$ be an arbitrary probability measure on $\mathscr{F}$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $\mathbb{Q} = \mathbb{P}$ on $\mathscr{F}_0$. Put $Z_t := \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathscr{F}_t\right])$, where $0 \leq t \leq T$. Then $Z_0 = 1$ $\mathbb{P}$-a.s., and both, Z and $\frac{1}{Z}$ have strictly positive paths $\mathbb{P}$-a.s.. In particular, Z and $\frac{1}{Z}$ can be represented as a stochastic exponential respectively, $\mathscr{L}(Z)$ exists, and $\Delta\mathscr{L}(Z) = \frac{1}{Z_-}\Delta Z > -1$ (outside some evanescent set). Moreover, Z is a uniformly integrable $\mathbb{P}$-martingale. If $M \in \mathscr{M}_{\text{loc}}(\mathbb{P})$, then $\widetilde{M} \in \mathscr{M}_{\text{loc}}(\mathbb{Q})$, where*

$$\begin{aligned}\widetilde{M} &:= M + \left[M, \mathscr{L}\left(\frac{1}{Z}\right)\right] \\ &= M - \left[M, \mathscr{L}(Z)\right]^c - \sum_{0<s\leq\bullet} \frac{\Delta Z_s}{Z_s} \Delta M_s\,.\end{aligned}$$

(The càdlàg version of) Z is known as the density process.

Proposition 2 *Let* $\mathbb{Q}$, Z *be as in Theorem 7. Then* $\mathscr{L}(Z)$ *is a Lévy process if and only if* $\log(Z)$ *is a Lévy process. Moreover,*

$$\nu^{\mathscr{L}(Z)} = \nu^{\log(Z)} \circ \psi^{-1},$$

where $\psi(y) := e^y - 1$ $(y \in \mathbb{R})$.

Theorem 7 immediately implies a generalisation of a result of Kazamaki which had been proven for continuous local martingales only (cf. Kazamaki (1994)):

Corrolary 3 (Girsanov Transformation) *Let* $\mathbb{Q}$ *and* Z *be as in Theorem 7. Then the mapping*

$$\begin{aligned} \Phi_{\mathbb{P},\mathbb{Q}} : \mathscr{M}_{\mathrm{loc}}(\mathbb{P}) &\stackrel{\simeq}{\longrightarrow} \mathscr{M}_{\mathrm{loc}}(\mathbb{Q}) \\ M &\longmapsto M + \left[M, \mathscr{L}\Big(\frac{1}{Z}\Big)\right] \end{aligned}$$

is an (algebraic) isomorphism, and $\Phi^{-1}_{\mathbb{P},\mathbb{Q}}(N) = N + [N, \mathscr{L}(Z)] \stackrel{!}{=} \Phi_{\mathbb{Q},\mathbb{P}}(N)$ *for any* $N \in \mathscr{M}_{\mathrm{loc}}(\mathbb{Q})$.

Problem: How do we choose 'good' density processes Z, i. e., 'good' ELMMs?

- Given an arbitrary (incomplete!) semimartingale market (S,N), any density process $Z_{\mathbb{P},\mathbb{Q}}$ – originating from any ELMM $\mathbb{Q} \sim \mathbb{P}$ – can be represented as a stochastic exponential:

$$Z_{\mathbb{P},\mathbb{Q}} = \mathscr{E}(X).$$

In accordance with the BMS case, we call $-X = -\mathscr{L}(Z_{\mathbb{P},\mathbb{Q}})$ market price of risk process.

- Interesting observation in the BMS case:

$$-\frac{\mu - r}{\sigma} W = \mathscr{L}\Big(Z_{\mathbb{P},\mathbb{Q}}\Big) \stackrel{!}{=} (g \circ \mathscr{L})\Big(\frac{S^B}{S_0}\Big),$$

where $g : \mathscr{S}_0^{\sim} \longrightarrow \mathscr{S}_0^{\sim}$ is given by

$$g(X)_t := -\frac{\mathbb{E}_{\mathbb{P}}[X_t]}{\mathrm{Var}_{\mathbb{P}}[X_t]}\,(X_t - \mathbb{E}_{\mathbb{P}}[X_t]).$$

Problem: Let (S,N) be as above and $\mathbb{Q}$ an arbitrary ELMM on $\mathscr{F}$. Then

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \stackrel{?}{=} \mathscr{E}\Big((\widetilde{g} \circ \mathscr{L})\Big(\frac{S^N}{S_0}\Big)\Big)_T.$$

How do we construct $\tilde{g}$? If we know (all) $\tilde{g}$, we know (all) $\mathbb{Q}$!

Definition 3 *Let* $M \in \mathscr{M}_{\text{loc},0}(\mathbb{P})$ *w.r.t.* $\mathbb{F}$. M *satisfies the strong property of predictable representation (SPPR) if the linear operator*

$$\begin{aligned} T : L_M(\mathbb{P}) &\longrightarrow\!\!\!\!\rightarrow \mathscr{M}_{\text{loc},0}(\mathbb{P}) \\ H &\longmapsto \int H dM \end{aligned}$$

is onto.

Theorem 8 (He/Wang/Yan (1992)) *Let* X *be a Lévy process and* $\mathbb{F} = \mathbb{F}^X$ *the augmented natural filtration of* X. *Assume that* X *is a (local) martingale. Then* X *satisfies the SPPR if and only if* X *is a standard Brownian motion or a compensated Poisson process, up to a constant factor.*

Theorem 9 *Let $M \in \mathscr{M}_{\text{loc},0}(\mathbb{P})$ satisfy the SPPR, and let $\mathbb{Q}$, Z be as in Theorem 7. Then there exists a predictable process H such that*

$$Z = \mathbb{E}_{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathscr{F}_{\bullet}\Big] = \mathscr{E}\Big(\int H dM\Big) = \exp(Y^H)$$

outside some evanescent set, where

$$Y_t^H := \int_0^t H_s dM_s - \frac{1}{2}\int_0^t H_s^2 d[M^c, M^c]_s + \Big(f * j_{\int HdM}\Big)_t.$$

Proof. $\widetilde{X} := \mathscr{L}(Z) \in \mathscr{M}_{\text{loc},0}(\mathbb{P})$. Since $Z = \mathscr{E}(\widetilde{X})$, the SPPR of M (applied to $\widetilde{X}$) implies the second equation. $X := \log(Z) = \log(\mathscr{E}(\widetilde{X}))$ is the logarithmic transform of $\widetilde{X} = \mathscr{L}(\exp(X))$. Now apply Lemma 2.6 of [**J. Kallsen, A. N. Shiryaev** (2002). *The cumulant process and Esscher's change of measure*. FS 6, 397-428]. □

Corrolary 4 *Let $\mathbb{Q}$ be as in Theorem 7. Then $\mathbb{E}_{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathscr{F}_{\bullet}^W\Big] = \mathscr{E}(\int HdW)$, where H is $\mathbb{F}^W$-predictable, and*

$$\widetilde{W}_t := W_t - \int_0^t H_s ds$$

defines a $\mathbb{Q}$-Brownian motion $\widetilde{W}$.

Corrolary 5 (Exponential Utility and Hedging) *Let* $M \in \mathscr{M}_{\text{loc},0}(\mathbb{P})$ *satisfy the SPPR, and let* $\mathbb{Q}$, H *be as in Theorem* 9.

$$\mathbb{E}_{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \alpha^H + \beta^H,$$

where

$$\alpha^H := \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T H_s\, dM_s - \frac{1}{2}\int_0^T H_s^2\, d[M^c, M^c]_s\Big]$$

and

$$\beta^H := \mathbb{E}_{\mathbb{Q}}\Big[\sum_{0<s\le T}\Big(\log\Big(\frac{Z_s}{Z_{s-}}\Big) - \frac{\Delta Z_s}{Z_{s-}}\Big)\Big].$$

Consequently,

$$\mathbb{E}_{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \alpha^H$$

if and only if M *is continuous.*

If in addition $\log(Z)$ resp. $\mathscr{L}(Z)$ is a Lévy process, we can calculate β^H more explicitly:

$$\frac{\beta^H}{T} = \int_{\mathbb{R}} f(x)\, \nu^{\mathscr{L}(Z)}(dx) = \int_{\mathbb{R}} (1+y-e^y)\, \nu^{\log(Z)}(dy),$$

where $f(x) := \log(x+1) - x$ $(x > -1)$.

Conclusion

Stochastic logarithms appear to be the key ingredients in financial mathematics! They do not only describe relative returns and link hedging with the calculation of minimal entropy martingale measures - and hence with utility-indifference. They even determine the structure of the Girsanov transformation.

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