Geometry of polar cones and superreplication prices in incomplete financial markets

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Contents

The market model

- We consider a security market with $d+1$ assets - one risk free asset (a bond) and d risky assets.

- $S \equiv (S^1, \ldots, S^d)$ denotes the discounted price process of the d assets, where each price is modelled as an arbitrary - *not necessarily locally bounded* - càdlàg semimartingale including jumps (e. g., a Lévy process) in a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with finite time horizon T.

- A *portfolio* is a pair $\Pi = (x, H)$, where

  - $x \in \mathbb{R}$ is the initial wealth;

  - $H \equiv (H^1, \ldots, H^d)$ is a trading strategy, i. e., a predictable and S-integrable $\mathbb{R}^d$-valued stochastic process, representing the holdings of the risky d assets.

Example 1 *Fix a scenario* $\omega \in \Omega$. *Assume you want to invest in one stock only, i. e.,* $d = 1$. *The only trading dates are* $0 = t_1 < \ldots < t_n = T$. *At any time point* t_{i-1} *you invest* $Z_{i-1}(\omega)$ *parts in the stock with price* $S_{t_{i-1}}(\omega)$. *The choice of* $Z_{i-1}(\omega)$ *is based on the information until* t_{i-1} *only!* No trading is then possible until t_i. *Thus, at* t_i, *your gain resp. loss is given by*

$$Z_{i-1}(\omega)\Big(S_{t_i}(\omega) - S_{t_{i-1}}(\omega)\Big).$$

In other words, if

$$H_t(\omega) := \mathbb{1}_{\{0\}}(t) Z_0(\omega) + \sum_{i=1}^{n} \mathbb{1}_{(t_{i-1},t_i]}(t) Z_{i-1}(\omega),$$

denotes your trading strategy H, *where each* Z_i *is a* $\mathscr{F}_{t_i}$*-measurable random variable, then your cumulative gain resp. loss process* $\int H \mathrm{d}S$ *is given by*

$$\Big(\int H \mathrm{d}S\Big)_t(\omega) := \sum_{i=1}^{n} Z_{i-1}(\omega)\Big(S_{t_i \wedge t}(\omega) - S_{t_{i-1} \wedge t}(\omega)\Big).$$

The admissible case

A trading strategy H is said to be *admissible* if the gains process is uniformly bounded from below by a (non-random) constant, i. e., if there exists a constant $c \geq 0$ such that for all $t \in [0, T]$,

$$G_t(H) := \int_0^t H_u \mathrm{d}S_u := \Big(\int H \mathrm{d}S\Big)_t \geq -c, \quad \mathbb{P}\text{–a. s.}.$$

This disallows doubling strategies: investors have a fixed credit limit.

A road to positivity The set of all attainable terminal wealths $G_T(H)$ from admissible trading strategies H with zero initial wealth is a convex cone (resp. wedge)

$$K^{\mathrm{adm}} := \Big\{G_T(H) : H \text{ is admissible}\Big\}$$

which is contained in the vector lattice $L^0 = L^0(\Omega, \mathscr{F}, \mathbb{P})$ of all random variables. Note that K^{adm} is not a subspace of L^0.

Let $X \in L^0$ be an arbitrary contingent claim. We assume that X satisfies the

Property (SR$^{\text{adm}}$) *There exists an admissible portfolio* $\Pi = (x, H)$ *which superreplicates* X*, i. e., we assume that*

$$\begin{aligned} A_X &:= \Big\{x \in \mathbb{R} : \exists H \text{ adm. s.t. } X \le x + G_T(H)\Big\} \\ &= \Big\{x \in \mathbb{R} : X - x \in K^{\text{adm}} - L^0_+\Big\} \neq \emptyset \,. \end{aligned}$$

Definition 1 *Assume that the set* $A_X \subseteq \mathbb{R}$ *is bounded from below. Then the superreplication price of* $X \in L^0$ *is defined by the following real number:*

$$\begin{aligned} \overline{\pi}(X) &:= \inf(A_X) \\ &= \inf\Big\{x \in \mathbb{R} : X - x \in K^{\text{adm}} - L^0_+\Big\}. \end{aligned}$$

When is the set A_X bounded from below? How do we then calculate $\overline{\pi}(X)$?

Separating probability measures

Consider

$$M_1^{\mathrm{adm}}(\mathbb{P}) := \left\{ \mathbb{Q} \ll \mathbb{P} : \mathbb{E}_{\mathbb{Q}}[G] \leq 0 \ \forall\, G \in K^{\mathrm{adm}} \right\}.$$

Remark 1 *Recall:*

(1) *If S is bounded then $M_1^{\mathrm{adm}}(\mathbb{P})$ is the set of all $\mathbb{P}$-absolutely continuous probability measures $\mathbb{Q}$ such that S is a $\mathbb{Q}$-martingale.*

(2) *If S is locally bounded then $M_1^{\mathrm{adm}}(\mathbb{P}) = M^a(\mathbb{P})$, where $M^a(\mathbb{P})$ denotes the set of all $\mathbb{P}$-absolutely continuous probability measures $\mathbb{Q}$ such that S is a $\mathbb{Q}$-local martingale.*

(3) *In the general non-locally bounded case: if $M_\sigma^a(\mathbb{P}) \neq \emptyset$ then $M_1^{\mathrm{adm}}(\mathbb{P})$ is the closure, in the topology induced by the total variation norm, of the set $M_\sigma^a(\mathbb{P})$ of all $\mathbb{P}$-absolutely continuous probability measures $\mathbb{Q}$ such that S is a $\mathbb{Q}$-sigma-martingale.*

Case 1: First, we reconsider the case of a contingent claim $X \in L^0$ which is bounded from below (for example, $X \geq 0$) and satisfies (SR$^{\text{adm}}$).

If S is locally bounded and $M^e(\mathbb{P}) \neq \emptyset$ then (cf. Delbaen-Schachermayer (1995))

$$\overline{\pi}(X) = \sup_{\mathbb{Q} \in M^e(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[X].$$

If S is a general semimartingale and $M^e_\sigma(\mathbb{P}) \neq \emptyset$ then (cf. Delbaen-Schachermayer (1998))

$$\overline{\pi}(X) = \sup_{\mathbb{Q} \in M^e_\sigma(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[X].$$

Case 2: Assume now that $M_1^{\text{adm}}(\mathbb{P}) \neq \emptyset$ and that $X \in L^0$ is arbitrary, hence not necessarily bounded from below. However, since we are interested in pricing the contingent claim X, it is natural to assume that $X \in L^1(\mathbb{Q})$ for any probability measure $\mathbb{Q} \in M_1^{\text{adm}}(\mathbb{P})$, hence that

$$X \in L(M_1^{\text{adm}}(\mathbb{P})) := \bigcap_{\mathbb{Q} \in M_1^{\text{adm}}(\mathbb{P})} L^1(\mathbb{Q}) \subseteq L^0.$$

In the following, let $M_1^{\mathrm{adm}}(\mathbb{P}) \neq \emptyset$.

Observation 1

$$K^{\mathrm{adm}} \subseteq L(M_1^{\mathrm{adm}}(\mathbb{P}))\,.$$

Hence, if $X \in L^0$ is bounded from below and satisfies (SR^{adm}) then $X \in L(M_1^{\mathrm{adm}}(\mathbb{P}))$.

Proof. Let $-r \leq X$. Then

$$|X| = |(X+r)-r| \leq (X+r)+|r| \leq (x+G_T(H))+2|r|.$$

□

Observation 2 *If $X \in L(M_1^{\mathrm{adm}}(\mathbb{P}))$ satisfies (SR^{adm}), then the set A_X is bounded from below and $\left\{\mathbb{E}_{\mathbb{Q}}[X] : \mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})\right\}$ is bounded from above. Moreover, we have*

$$-\infty < \sup_{\mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[X] \leq \overline{\pi}(X) < +\infty.$$

Proof. Let $x \in \mathbb{R}$ such that $X - x \in K^{\mathrm{adm}} - L^0_+$ and $\mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})$ arbitrarily chosen. Then

$$-\infty < -\mathbb{E}_{\mathbb{Q}}[|X|] \leq \mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{Q}}[X-x]+x \leq 0+x\,.$$

□

However, one can construct $Y \in L(M_1^{\mathrm{adm}}(\mathbb{P}))$ (cf. Biagini-Frittelli (2004)) such that

$$\sup_{\mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[Y] < \overline{\pi}(Y).$$

Therefore the (convex) cone K^{adm} of admissible trading strategies seems to be unsuitable for the superreplication of unbounded claims! Intuitively, the cone K^{adm} is too small.

To reveal the underlying – geometric – key ideas how to maintain equality even in the unbounded and (later) in the non-admissible case, recall that

$$K^{\mathrm{adm}} \subseteq L(M_1^{\mathrm{adm}}(\mathbb{P})) = \bigcap_{\mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})} L^1(\mathbb{Q}).$$

Let $X \in L(M_1^{\mathrm{adm}}(\mathbb{P}))$ such that X satisfies (SR$^{\mathrm{adm}}$). Let C be an arbitrary cone such that

$$K^{\mathrm{adm}} \subseteq C \subseteq L(M_1^{\mathrm{adm}}(\mathbb{P}))$$

and

$\overline{\pi}(X;C) := \inf\left\{x \in \mathbb{R} : X \leq x + G \text{ for some } G \in C\right\}$
exists. Then

$$\sup_{\mathbb{Q}\in M_1^{\mathrm{adm}}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[X] \overset{?}{\leq} \overline{\pi}(X;C) \overset{\checkmark}{\leq} \overline{\pi}(X;K^{\mathrm{adm}}) = \overline{\pi}(X)\,.$$

Consequently, we have arrived at the following general problem which even will allow us to evade the too restrictive admissibility condition:

Problem: *Fix an arbitrary cone K in L^0 and a suitable (convex) set of pricing measures $M_1(\mathbb{P};K)$. Do there exist a non-empty set $M \subseteq M_1(\mathbb{P};K)$ and a sufficiently large cone $C \subseteq \bigcap_{\mathbb{Q}\in M} L^1(\mathbb{Q}) =: L(M)$ such that $K \subseteq C$ and*

$$\sup_{\mathbb{Q}\in M} \mathbb{E}_{\mathbb{Q}}[X] = \overline{\pi}(X;C)\ \forall\, X \in L(M)\,?$$

To solve this problem, we now concentrate our attention on a suitable class of cones in vector lattices.

Definition 2 *Let* $(E, \leq)$ *be a vector lattice and* $C \subseteq E$ *a cone in* E. C *is an umbrella (cone) in* E *if* $-E_+ = \{x \in E : x \leq 0\} \subseteq C$.

Given an arbitrary cone $K \subseteq E$, we denote by $\mathsf{s}_E(K)$ the umbrella hull, i. e., the smallest umbrella in E which contains K. Note that

$$\mathsf{s}_E(K) = \left\{x \in E : \exists\, g \in K \text{ s. t. } x \leq g\right\} = K - E_+.$$

Consequently, a cone $C \subseteq E$ is an umbrella in E if and only if $C = \mathsf{s}_E(C) = C - E_+$.

Proposition 1 *Let* $(E_\alpha)_{\alpha \in A}$ *be a family of vector sublattices of a vector lattice* L. *Then* $E := \bigcap_{\alpha \in A} E_\alpha$ *is a vector sublattice of* L. *Let* $K \subseteq E$ *be an arbitrary cone in* E. *Then*

$$s_E(K) = \bigcap_{\alpha \in A} s_{E_\alpha}(K) = \bigcap_{\alpha \in A} \left(K - (E_\alpha)_+\right).$$

Example 2 *Assume that* $M_1^{\mathrm{adm}}(\mathbb{P}) \neq \emptyset$. *Set* $E := L(M_1^{\mathrm{adm}}(\mathbb{P}))$. *Then* $K^{\mathrm{adm}} \subseteq E$, *and*

$$s_E(K^{\mathrm{adm}}) = \bigcap_{\mathbb{Q} \in M_1^{\mathrm{adm}}(\mathbb{P})} \left(K^{\mathrm{adm}} - L_+^1(\mathbb{Q})\right).$$

Superreplication prices: the general approach

We now extend the definition of the superreplication price to allow terminal wealths from an arbitrary cone K in L^0.

(i) Fix an arbitrary cone $K \subseteq L^0$.

(ii) Set $M_1(\mathbb{P}; K) := \Big\{ \mathbb{Q} \ll \mathbb{P} : K \subseteq L^1(\mathbb{Q})$ and $\mathbb{E}_{\mathbb{Q}}[G] \leq 0$ for all $G \in K \Big\}$.

(iii) Consider an arbitrary non-empty subset M of $M_1(\mathbb{P}; K)$ and set $L(M) := \bigcap_{\mathbb{Q} \in M} L^1(\mathbb{Q})$. Note that by construction

$$K \subseteq L(M_1(\mathbb{P}; K)) \subseteq L(M)\,.$$

(iv) Let $X \in L(M)$ and $C \subseteq L(M)$ be an arbitrary cone such that $K \subseteq C$. Assume that $A_X(K) \neq \emptyset$, where

$$A_X(C) := \{x \in \mathbb{R} : \exists\, G \in C \text{ s.t. } X \leq x + G\}.$$

$A_X(K)$ is bounded from below and we have $A_X(K) \subseteq A_X(C) = A_X\big(\mathsf{s}_{L(M)}(C)\big)$. If $A_X(C)$ is bounded from below, we set

$$\overline{\pi}(X;C) := \inf(A_X(C)) \leq \overline{\pi}(X;K).$$

Similarly to **Observation 2**, it follows that

$$-\infty < \sup_{\mathbb{Q} \in M} \mathbb{E}_{\mathbb{Q}}[X] \leq \overline{\pi}(X;K) < +\infty.$$

Moreover,

$$\begin{aligned} \overline{\pi}(X;C) &= \inf\Big\{x \in \mathbb{R} : X - x \in \mathsf{s}_{L(M)}(C)\Big\} \\ &= \inf\Big\{x \in \mathbb{R} : X - x \in \bigcap_{\mathbb{Q} \in M} \Big(C - L^1_+(\mathbb{Q})\Big)\Big\}. \end{aligned}$$

Thus, we have arrived at the following natural question: *Given an arbitrary cone $K \subseteq L^0$, which cones $C \supseteq K$ and which non-empty sets M of separating probability measures would then be suitable to solve our pricing problem? Would then $\mathsf{s}_{L(M)}(K)$ be a subset of C?*

Pricing measures and utility functions I

First answers were provided by Biagini and Frittelli (for K^{adm} only!) and Oe.-Owen (already for arbitrary cones $K \subseteq L^0$), where preferences of an investor were incorporated in the construction of an enlarged cone in the space $L(M_\Phi)$ by means of the Fenchel-Legendre conjugate Φ of the investor's utility function U. Given an arbitrary cone $K \subseteq L^0$, this cone was defined as

$$C_\Phi := \bigcap_{\mathbb{Q} \in M_\Phi} \overline{K - L^1_+(\mathbb{Q})}^{L^1(\mathbb{Q})},$$

where

$$M_\Phi \equiv M_\Phi(K) := \left\{ \mathbb{Q} \in M_1(\mathbb{P}; K) \,\middle|\, \Phi \circ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}) \right\}$$

(which is the set of all pricing measures with finite entropy if $K := K^{\text{adm}}$). If $M_\Phi \neq \emptyset$, then

$$\mathrm{s}_{L(M_\Phi)}(K) = \bigcap_{\mathbb{Q} \in M_\Phi} \left(K - L^1_+(\mathbb{Q}) \right) \subseteq C_\Phi \subseteq L(M_\Phi).$$

If $M_\Phi \equiv M_\Phi(K^{\text{adm}})$, the following result holds:

Theorem 1 (Biagini-Frittelli (2004)) *Let $U : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly concave, strictly increasing C^1-utility function, satisfying the Inada conditions and Reasonable Asymptotic Elasticity at $-\infty$ and $+\infty$. Assume that $M_\Phi^{\text{adm}} \neq \emptyset$ and $\Phi(0) < +\infty$, where Φ denotes the Fenchel-Legendre conjugate of U and let $X \in L(M_\Phi^{\text{adm}})$. Then*

$$\overline{\pi}(X; C_\Phi^{\text{adm}}) = \sup_{\mathbb{Q} \in M_\Phi^{\text{adm}}} \mathbb{E}_{\mathbb{Q}}[X].$$

By slightly enlarging the set M_Φ, we will recognise that we only need Reasonable Asymptotic Elasticity of U at $-\infty$ (but not at $+\infty$) and that we can drop the boundedness assumption $\Phi(0) < +\infty$.

The proof of this generalisation is built on a duality theory for (convex) cones which does not only allow a transfer of Farkas' Lemma (i. e., dual LPs) to infinite-dimensional normed lattices. It also gives us a very helpful bipolar relation where none was possible before.

Duality of polar cones

Let (E, F) be an arbitrary bilinear system of vector spaces over $\mathbb{R}$. For a non-empty set $A \subseteq E$ the polar of A is often defined in two different ways in the literature:

Polar I:

$$A^{\circ} := \Big\{ w \in F : \langle z, w \rangle \leq 1 \;\; \forall z \in A \Big\};$$

Polar II:

$$A^{\bullet} := \Big\{ w \in F : |\langle z, w \rangle| \leq 1 \;\; \forall z \in A \Big\}.$$

Note that $A^{\bullet} \subseteq A^{\circ}$. Both definitions coincide if A is circled (i. e., if $[-1, 1] \cdot A \subseteq A$).

In addition, we introduce the polar cone of A:

$$A^{\triangleleft} := \{ w \in F : \langle z, w \rangle \leq 0 \;\; \forall z \in A \}.$$

Clearly, $A^{\triangleleft}$ is a cone in F and $A^{\triangleleft} \subseteq A^{\circ}$.

Similarly, for a non-empty set $B \subseteq F$, we define its polar cone $B^{\triangleleft} \subseteq E$.

Let $\emptyset \neq A \subseteq E$. Let cone($A$) be the smallest cone in E which contains A.

Proposition 2 *Let (E, F) be an arbitrary bilinear system of real vector spaces and A be an arbitrary non-empty subset of E. Then:*

$$A^{\triangleleft} = (\text{cone}(A))^{\circ}.$$

In particular, $A^{\triangleleft}$ is $\sigma(F, E)$-closed.

Corollary 1 *Let (E, F) be an arbitrary bilinear system of real vector spaces and A be an arbitrary non-empty subset of E. Then:*

$$A^{\triangleleft\triangleleft} = \overline{\text{cone}(A)}^{\sigma(E,F)}.$$

In other words, $A^{\triangleleft\triangleleft}$ is the smallest $\sigma(E, F)$-closed cone in E which contains A. If in addition A is a cone, then $A^{\triangleleft\triangleleft} = \overline{A}^{\sigma(E,F)}$.

Let K be an arbitrary cone in L^0. Recall the set

$$M_1(\mathbb{P};K) := \Big\{\mathbb{Q} \ll \mathbb{P} : K \subseteq L^1(\mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}[G] \leq 0 \text{ for all } G \in K\Big\}.$$

Let $\emptyset \neq M \subseteq M_1(\mathbb{P};K)$. By identifying probability measures $\mathbb{Q} \ll \mathbb{P}$ with their Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1_+(\mathbb{P})$ of norm 1, we set

$$E := L(M) := \bigcap_{\mathbb{Q}\in M} L^1(\mathbb{Q}) \supseteq K$$

and

$$F := \text{Lin}(M) \overset{\checkmark}{\subseteq} L^1(\mathbb{P}).$$

Hence, for any pair $(Z,W) \in E \times F$, the following dual pairing is well-defined:

$$\langle Z, W\rangle := \mathbb{E}_{\mathbb{P}}[ZW].$$

In particular,

$$\langle Z, \mathbb{Q}\rangle \equiv \langle Z, \frac{d\mathbb{Q}}{d\mathbb{P}}\rangle = \mathbb{E}_{\mathbb{Q}}[Z] \;\forall \mathbb{Q} \in M.$$

A standard exercise in measure theory shows that (E, F) is a well-defined left-dual (bilinear) system. Consequently,

$$F = (E, \tau)',$$

where $\tau := \sigma(E, F)$. To understand the following crucial result, recall that

$$\mathsf{s}_E(K) = \bigcap_{\mathbb{Q} \in M} (K - L^1_+(\mathbb{Q})) = K - E_+ \subseteq E$$

describes the umbrella hull of the cone K in the vector sublattice E of L^0.

Proposition 3 *Let K be an arbitrary cone in L^0, and let $\emptyset \neq M \subseteq M_1(\mathbb{P}; K)$. Then*

$$L^1_+(\mathbb{P}) \cap K^{\triangleleft} = F \cap \operatorname{cone}(M_1(\mathbb{P}; K)) = (\mathsf{s}_E(K))^{\triangleleft}.$$

In particular, the set $F \cap \operatorname{cone}(M_1(\mathbb{P}; K))$ is $\sigma(F, E)$-closed, and it contains $\overline{\operatorname{cone}(M)}^{\sigma(F,E)}$.

Corollary 2 *Let* K *be an arbitrary cone in* L^0 *and let* $\emptyset \neq M \subseteq M_1(\mathbb{P};K)$. *Set* $C_E(K) := \overline{s_E(K)}^{\sigma(E,F)}$. *TFAE:*

(i) $F \cap \mathrm{cone}(M_1(\mathbb{P};K)) = \overline{\mathrm{cone}(M)}^{\sigma(F,E)}$;

(ii) $M^{\lhd} = C_E(K)$;

(iii) $\overline{\mathrm{cone}(M)}^{\sigma(F,E)} = (C_E(K))^{\lhd} = (s_E(K))^{\lhd}$.

Example 3 *Set* $E_1 := L(M_1(\mathbb{P};K))$ *and* $F_1 := \mathrm{Lin}(M_1(\mathbb{P};K))$. *If* $M_1(\mathbb{P};K) \neq \emptyset$, *Proposition* 3 *implies that*

$$\mathrm{cone}(M_1(\mathbb{P};K)) = (C_{E_1}(K))^{\lhd}$$

is $\sigma(F_1,E_1)$*-closed. Moreover,*

$$M_1(\mathbb{P};K)^{\lhd} = C_{E_1}(K)\,.$$

Theorem 2 *Let K be an arbitrary cone in L^0, and let $\emptyset \neq M \subseteq M_1(\mathbb{P}, K)$ such that*

$$\overline{\text{cone}(M)}^{\sigma(F,E)} = F \cap \text{cone}(M_1(\mathbb{P}, K)).$$

Then $K \subseteq C_E(K) = M^{\triangleleft} \subseteq E$. Let $X \in E$. Then the set $A_X(C_E(K))$ is bounded from below and

$$\overline{\pi}(X; C_E(K)) = \sup_{\mathbb{Q} \in M} \mathbb{E}_{\mathbb{Q}}[X] .$$

Proof. Since the cone $C_E(K)$ obviously is an umbrella in E, it follows that

$$\begin{aligned} \overline{\pi}(X; C_E(K)) &= \inf\{x \in \mathbb{R} : X - x \in C_E(K)\} \\ &= \inf\{x \in \mathbb{R} : X - x \in M^{\triangleleft}\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{E}_{\mathbb{Q}}[X - x] \leq 0 \, \forall \, \mathbb{Q} \in M\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{E}_{\mathbb{Q}}[X] \leq x \, \forall \, \mathbb{Q} \in M\} \\ &= \sup_{\mathbb{Q} \in M} \mathbb{E}_{\mathbb{Q}}[X] . \end{aligned}$$

$\square$

Corollary 3 *Let K be an arbitrary cone in L^0. Set* $E_1 := \bigcap_{\mathbb{Q}\in M_1(\mathbb{P};K)} L^1(\mathbb{Q}) = L(M_1(\mathbb{P};K))$ *and* $F_1 := \mathrm{Lin}(M_1(\mathbb{P};K))$*. Then*

$$\overline{\pi}(X; C_{E_1}(K)) = \sup_{\mathbb{Q}\in M_1(\mathbb{P},K)} \mathbb{E}_{\mathbb{Q}}[X] \;\forall\, X \in E_1 .$$

Consequently, due to Biagini-Frittelli's counter-example, the (convex) cone $\mathsf{s}_{E_1^{\mathrm{adm}}}(K^{\mathrm{adm}})$ is not $\sigma(E_1^{\mathrm{adm}}, F_1^{\mathrm{adm}})$-closed.

Pricing measures and utility functions II

Let K be an arbitrary cone in L^0.

Assumption 1 *We assume that the investor has a critical wealth* $a \in \{-\infty\} \cup \mathbb{R}$ *and a utility function* $U : (a, \infty) \to \mathbb{R}$ *which is increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions*

$$\lim_{x \downarrow a} U'(x) = \infty, \qquad \lim_{x \uparrow \infty} U'(x) = 0.$$

Furthermore, if the domain of U *is the whole real line (i. e., if* $a = -\infty$ *) then we assume that* U *has Reasonable Asymptotic Elasticity at* $-\infty$*, in the sense that*

$$AE_{-\infty}(U) := \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1.$$

The Fenchel-Legendre conjugate Φ of the utility function U is defined for $y > 0$ by

$$\Phi(y) := \sup_{x \in (a,\infty)} (U(x) - xy).$$

In the following, we set $M_1 := M_1(\mathbb{P}, K)$. Note that

$$M_\Phi = \left\{ \mathbb{Q} \in M_1 \,\middle|\, \mathbb{E}_{\mathbb{P}}\left[\Phi^+\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < \infty \right\}.$$

Definition 3 *A measure $\mathbb{Q} \ll \mathbb{P}$ is said to have finite loss-entropy if there exists a constant $b > 0$ such that*

$$\mathbb{E}_{\mathbb{P}}\left[\Phi^+\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \mathbb{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \geq b\}}\right] < \infty.$$

The set of separating measures with finite loss-entropy is therefore given by

$$\widehat{M_\Phi} := \left\{ \mathbb{Q} \in M_1 \,\middle|\, \mathbb{E}_{\mathbb{P}}\left[\Phi^+\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \mathbb{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \geq b\}}\right] < \infty \right\}.$$

Clearly, the definition of the set $\widehat{M_\Phi}$ does not depend on the choice of the constant $b > 0$.

Since our machinery is set up, we only have to adapt the previous definitions. Assume throughout that $\widehat{M_\Phi} \neq \emptyset$. Our left-dual system is now given by

$$E_\Phi := L(\widehat{M_\Phi}) := \bigcap_{\mathbb{Q} \in \widehat{M_\Phi}} L^1(\mathbb{Q})$$

and

$$F_\Phi := \text{Lin}(\widehat{M_\Phi}).$$

Set $C_\Phi := C_{E_\Phi}(K)$. It turns out that cone(M_Φ) might not be $\sigma(F_\Phi, E_\Phi)$-closed. However,

Theorem 3

$$\text{cone}(\widehat{M_\Phi}) = F_\Phi \cap \text{cone}(M_1(\mathbb{P}; K)).$$

Consequently, cone$(\widehat{M_\Phi})$ is $\sigma(F_\Phi, E_\Phi)$-closed and the pair $\left(C_\Phi, \text{cone}(\widehat{M_\Phi})\right)$ satisfies a "symmetric cone duality".

Corollary 4

$$\overline{\pi}(X; C_\Phi) = \sup_{\mathbb{Q} \in \widehat{M_\Phi}} \mathbb{E}_{\mathbb{Q}}[X] \ \forall\, X \in E_\Phi.$$

The next result links our approach with the very useful theory of Orlicz spaces:

Observation 3 *Let $a = -\infty$. Assume that in addition to Assumption 1 Φ is of "De La Vallée Poussin" type, i. e., assume that*

$$\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty \,,$$

then Φ satisfies the Δ_2 condition:

$$\exists b > 0 \; \forall \alpha > 1 \; \exists k_\alpha > 0 : \Phi(\alpha y) \le k_\alpha \Phi(y) \; \forall y > b \,.$$

There is a very nice economic interpretation of C_Φ: Since $C_\Phi \overset{(!)}{=} (\widehat{M_\Phi})^\triangleleft$ *depends only on the finite loss-entropy measures*, it depends only on the values of the Fenchel-Legendre conjugate $\Phi(y)$ for arbitrarily large values of y, and hence on values of the utility function $U(x)$ for arbitrarily large negative values of x. So the only important information is contained in the preferences of the investor to asymptotically large losses.

Note also that the utility preferences of the investor to asymptotically large losses are encoded in the spaces E_Φ and F_Φ.

A very small extract of references

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