A Hilbert space approach to Wiener Chaos Decomposition and applications to finance

Work in Progress

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Abstract

The Malliavin calculus (also known as the stochastic calculus of variations) is an infinite–dimensional differential calculus on the Wiener space, initiated by Malliavin and further developed by Bismut, Stroock and others (cf. [9], [2], [14]) and which has been shown as a very important tool for mathematical physicists working in Quantum Field Theory.

We consider the first aspects of this calculus which have been used in financial applications, such as the pricing and hedging of path dependent options or the investigation of residual risk in incomplete markets. To be able to understand these applications, we have to work through the theory and methods of the underlying mathematical machinery. In this paper, we use only some elementary Hilbert space methods to obtain another description of the multiple Wiener integral (given a finite time horizon $T > 0$ and the natural filtration induced by the Wiener process W) and its relations to a decomposition of an arbitrary random variable in $L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ in successive Wiener chaoses, which works in a similar way as a Taylor's expansion but taking into account the stochastic dynamics. In particular we will present the main ideas, leading to this representation of $L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$. We recognize the appearance of the well known family of stochastic exponentials and its relations to the Itô representation of such L^2–random variables.

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given a (one-dimensional) Wiener process W and its natural filtration $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t \leq T}$, let us recall the standard example in the classical

Black & Scholes-market (without transaction costs), where the stock price S is modelled as $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$ and the price of the riskless asset is given by $dB_t = rB_t \, dt$, where μ, $\sigma \in \mathbb{R}$, $\sigma \neq 0$ and $r \geq 0$. Then it is well known that there exists an eqivalent martingale measure (EMM) $Q \sim \mathbb{P}$ so that $B_t := W_t + \frac{\mu - r}{\sigma} t$ defines a standard Q-Wiener Process B (so that $\mathbb{F}^W = \mathbb{F}^B$) and $\widetilde{S}_t := \exp(-rt) \, S_t$ defines a Q-martingale $\widetilde{S}$, which implies that *every* contingent claim $Z \in L^2(\Omega, \mathcal{F}_T^B, Q)$ can be replicated by a self-financing portfolio and that the value of any replicating self-financing portfolio Π at time t is given by

$$\Pi_t = \mathbb{E}_Q(e^{-r(T-t)} Z \mid \mathcal{F}_t^B) \ .$$

The existence of such a replicating portfolio follows by the Kunita-Watanabe representation theorem - applied to the the $Q-$square-integrable martingale Π (cf. [7]):

Theorem 1.1 *Let $M = (M_t)_{0 \leq t \leq T}$ be a square-integrable martingale, with respect to the filtration $\mathbb{F}^B$. Then there exists an adapted process K so that $\mathbb{E}_Q(\int_0^T K_s^2 \, ds) < \infty$ and*

$$M_t = M_0 + \int_0^t K_s \, dB_s$$

for all $t \in [0, T]$.

Corrolary 1.1 *Let $Z \in L^2(\Omega, \mathcal{F}_T^B, Q)$. Then there exists an adapted process L so that $\mathbb{E}_Q(\int_0^T L_s^2 \, ds) < \infty$ and*

$$Z = \mathbb{E}_Q(Z) + \int_0^T L_s \, dB_s \ .$$

Unfortunately, this is only a statement leading to the *existence* of a hedging strategy $H := K / \sigma \widetilde{S}$, which in general does not tell us, *how* this portfolio looks like. However, if the contingent claim Z can be written as $Z = f(S_T)$, (e.g., if Z is a standard Call option) we can express the portfolio value Π_t as a sufficiently smooth function of t and S_t, $\Pi_t = F(t, S_t)$, and an application of Itô's formula leads to the concrete value $H_t = \frac{\partial F}{\partial x}(t, S_t)$ - satisfying trader's needs. Here, Malliavin calculus enters the scene. Due to this calculus, it is possible to transfer the Kunita-Watanabe representation to random variables which are functionals of Brownian motion. It turns out that the process L in the previous corollary can be identified as the optional projection of the Malliavin *derivative* process $(D_t F)_{0 \leq t \leq T}$ *of* F. In these lecture notes we finally explain the construction of such an derivative[1] and prove and formulate precisely the following (cf. [5])

Theorem 1.2 *Let $Z = F(B)$ be a functional of the Brownian motion B. Then, under technical hypotheses on F*

$$Z = \mathbb{E}_Q(Z) + \int_0^T \mathbb{E}_Q(D_s F \mid \mathcal{F}_s^B) \, dB_s \ .$$

[1]This means, that we want to "differentiate" F with respect to the chance variable $\omega \in \Omega$ - without assuming any topological structure on Ω.

This result, built on deep theorems in Malliavin calculus, directly lead to applications in finance, such as the *construction* of hedging portfolios of certain exotic options or the investigation of discontinuous path-dependent payoff functionals of multidimensional diffusion processes (cf. [1], [3], [4]). A general approach to the analysis of dynamic hedging portfolos is discussed in [8]. The common source of these papers (and the basics of the Malliavin calculus) is the nontrivial fact that option prices can be decomposed in Wiener chaoses. This chaotic decomposition works in a similar way as a Taylor's power series expansion in complex analysis, taking additionally into account the stochastic dynamics of the underlying asset price process. As in complex analysis, we will see, that the Malliavin derivative allows us to differentiate piecewise "power series" (in $L^2(\Omega)$), and we will obtain the Itô integral as a special case of the Skorohod integral, which is the adjoint of the Malliavin derivative operator. Since the Wiener chaos decomposition plays the crucial role in the Malliavin calculus, we give a detailed introduction to it, including complete proofs. We only use elementary facts of Hilbert space theory. Nevertheless, our (analytic) proofs could lead to a transfer to more general semimartingales than Brownian motion.

2 The construction of the multiple Wiener integral

Let $T > 0$ fixed and $n \in \mathbb{N}$. Let $f \in L^2(S_n(T))$, where $S_n(T) := \{(t_1, t_{2,...,} t_n) \in [0,T]^n : t_1 \leq t_2 \leq ...t_n\}$ denotes the $n-$ simplex in the cube $[0,T]^n$. Let W be a Wiener process (Brownian motion) and $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t \leq T}$ the usual W-augmented filtration. We want to construct an *iterated Itô-integral*, in the following sense

$$J_n f = \int_0^T (\int_0^{t_n} ...(\int_0^{t_2} f(t_1, t_2, ..., t_n) dW_{t_1})...dW_{t_{n-1}}) dW_{t_n} \ . \tag{1}$$

To prove and understand the existence of (1), we use the well known Hilbert space approach to the Itô-integral. So let us recall the basic Hilbert spaces which lead to the fundamental *Itô isometry:* $L^2(\Omega \times [0,T], \mathcal{P}^W, \lambda_T \otimes \mathbb{P}) =: L_T^2(W)$ is the Hilbert space of all predictable processes X with $\|X\|^2_{L_T^2(W)} = \mathbb{E}(\int_0^T X_s^2 ds) < \infty$, where $\mathcal{P}^W$ denotes the predictable $\sigma-$algebra, i.e., the $\sigma-$algebra, generated by all left-continuous and $\mathbb{F}^W$-adapted processes. Since W is (left-)continuous, $\mathbb{F}^W$-adapted processes and $\mathbb{F}^W$-predictable processes coincide. If $X \in L_T^2(W)$, then it is well known, that the stochastic integral $X \bullet W = (\int_0^t X_s \; dW_s)_{0 \leq t \leq T}$ is a $L^2(\Omega)$-bounded (and continuous) martingale: $X \bullet W \in M_T^2(W) := \{Z : Z$ is a $\mathbb{F}^W$-martingale and $\sup_{0 \leq s \leq T} \mathbb{E}(Z_s^2) < \infty\}$. Thus, it is an uniformly integrable and (in particular) closable martingale with closing element $(X \bullet W)_T = \int_0^T X_s dW_s \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$. Let us recall the following

Theorem 2.1 (Itô Isometry) $L_T^2(W) \overset{1}{\hookrightarrow} L^2(\Omega, \mathcal{F}_T^W, \mathbb{P}),\ X \mapsto (X \bullet W)_T$ *is an isometric linear embedding:*

$$\mathbb{E}((X \bullet W)_T^2) = \mathbb{E}(\int_0^T X_s^2 ds).$$

Let $L_0^2(\Omega, \mathcal{F}_T^W, \mathbb{P}) := \{X \in L^2(\Omega) : \mathbb{E}(X) = 0\}$. Then $L_0^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ obviously is a subspace of the Hilbert space $L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$, and the Kunita-Watanabe representation theorem, together with the Itô isometry now reveal the small "size" of the space $L_T^2(W)$ of all square integrable $\mathbb{F}^W$-adapted processes:

Theorem 2.2 $L^2_T(W) \cong L^2_0(\Omega, \mathcal{F}^W_T, \mathbb{P})$.

To construct the iterated Itô integral recursively, we first take a closer look at the finite dimensional simplexes. So, let $k \geq 2$ be an arbitrary natural number and $t > 0$. Given a function $g : \mathbb{R}^k \longrightarrow \mathbb{R}$, we set $g(\cdot, s)(u) := g(u, s)$, where $u \in \mathbb{R}^{k-1}$ and $s \in \mathbb{R}$. Since

$$\mathbb{1}_{S_k(t)}(u, s) = \mathbb{1}_{S_{k-1}(s)}(u) \cdot \mathbb{1}_{[0,t]}(s)$$

for all $(u, s) \in S_k(t)$, Fubini's theorem directly leads to the following

Lemma 2.1 *If $g \in L^2(S_k(t))$, then $g(\cdot, s) \in L^2(S_{k-1}(s))$ for all $s \in [0, t]$, and*

$$\int\limits_{S_k(t)} g(v)\, d^k v = \int\limits_0^t \Big(\int\limits_{S_{k-1}(s)} g(u, s)\, d^{k-1} u \Big)\, ds\,.$$

The above isometry and Fubini-allowed interchanging of the operators $\mathbb{E}$ and $\int_0^t$, now lead us to a precise meaning of the iterated Itô-integral. Given an arbitrary $t > 0$, we will define linear isometries $Y^{(k)} : L^2(S_k(t)) \to L^2_t(W)$ recursively by[2]

$$Y^{(1)} g(\omega, s) := g(s) \quad (s \in [0, t]) \qquad \text{[deterministic]}$$

and

$$Y^{(k)} g(\omega, s) := (Y^{(k-1)} g(\cdot, s) \bullet W)_s(\omega) \quad (s \in [0, t])$$

for $k > 1$. The next proposition will show us that these operators are well defined.

Proposition 2.1 *Let $k, l \in \mathbb{N}$ and $t > 0$. Then $Y^{(k)} : L^2(S_k(t)) \to L^2_t(W)$ defines a linear isometry from the Hilbert space $L^2(S_k(t))$ into the Hilbert space $L^2_t(W)$ and, if $k \neq l$, then*

$$(Y^{(k)} g \mid Y^{(l)} h)_{L^2_t(W)} = 0 \ \text{ for all } g \in L^2(S_k(t)) \text{ and } h \in L^2(S_l(t)). \tag{2}$$

Proof: Nothing is to show for $k = 1$. So let $g \in L^2(S_{k+1}(t))$ given and the linear isometry $Y^{(k)} : L^2(S_k(s)) \to L^2_s(W)$ already be constructed (for arbitrary $s \in [0, T]$). Let $s \in [0, t]$. Fubini's theorem implies that $g(\cdot, s) \in L^2(S_k(s))$, so that $Y^{(k)} g(\cdot, s) \bullet W \in M^2_s(W)$ exists. Hence, setting $(Y^{(k+1)} g)_s := (Y^{(k)} g(\cdot, s) \bullet W)_s = \int_0^s (Y^{(k)} g(\cdot, s))_u\, dW_u \in L^2(\Omega, \mathcal{F}^W_s, \mathbb{P})$, the previous considerations show that

$$\mathbb{E}(\int_0^t (Y^{(k+1)} g)^2_s\, ds) = \int_0^t \mathbb{E}((Y^{(k+1)} g)^2_s)\, ds = \int_0^t (\mathbb{E}(\int_0^s (Y^{(k)} g(\cdot, s))^2_u)\, du)\, ds.$$

By assumption, $\mathbb{E}(\int_0^s (Y^{(k)} g(\cdot, s))^2_u)\, du = \int_{S_k(s)} g^2(u, s)\, d^k u$, and it follows that

$$\|Y^{(k+1)} g\|^2_{L^2_t(W)} = \int_{S_{k+1}(t)} g^2(v)\, d^{k+1} v = \|g\|^2_{L^2(S_{k+1}(t))} < \infty.$$

Now we will prove (2). Given $m \in \mathbb{N}$, we have to show that $(Y^{(k)} g \mid Y^{(k+m)} h)_{L^2_t(W)} = 0$ for all $k \in \mathbb{N}$. First, let $k = 1$. Then $(Y^{(1)} g \mid Y^{(1+m)} h)_{L^2_t(W)}$

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[2]More precisely, we should use the symbol $Y^{(k)}_t$ to denote the dependence on the parameter t.

$= \mathbb{E}(\int_0^t g(s) \cdot (Y^{(1+m)}h)_s \; ds) = \int_0^t g(s) \cdot \mathbb{E}((Y^{(1+m)}h)_s) \; ds = 0$, since $Y^{(1)}g = g$ is *deterministic* and $Y^{(m)}h(\cdot, s) \bullet W$ is a martingale which starts at 0. Let $g \in L^2(S_{k+1}(t))$ and $h \in L^2(S_{k+1+m}(t))$. Then, given the assumption that (2) is valid for k, the Itô isometry implies that in particular

$$\begin{aligned} \mathbb{E}((Y^{(k+1)}g)_s \cdot (Y^{(k+1+m)}h)_s) &= ((Y^{(k+1)}g)_s \mid (Y^{(k+1+m)}h)_s)_{L^2(\Omega, \mathcal{F}_s^W, \mathbb{P})} \\ &= (Y^{(k)}g(\cdot, s) \mid Y^{(k+m)}h(\cdot, s))_{L_s^2(W)} \\ &= 0 \end{aligned}$$

for all $s \in [0, t]$. Integrating both sides over $[0, t]$, finishes the proof. □

Now we define for arbitrary $T > 0$, $n \in \mathbb{N}$ and $f \in L^2(S_n(T))$

$$J_n f := (Y^{(n)} f \bullet W)_T = \int_0^T (Y^{(n)} f)_s \, dW_s \; .$$

Evaluating this expression recursively, it follows by construction that

$$J_n f = \int_0^T J_{n-1} f(\cdot, s) \; dW_s$$

for all $n > 1$, and we obtain the above expression (1). Setting $J_0 c := c$ for $c \in L^2(S_0(T)) := \mathbb{R}$, we arrive at the following

Theorem 2.3 *Let $n \in \mathbb{N}_0$ and $T > 0$. Then*

$$J_n : L^2(S_n(T)) \overset{1}{\hookrightarrow} L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$$

is a linear isometry, and if $m \in \mathbb{N}_0$ with $m \neq n$, then

$$\mathbb{E}(J_n f \cdot J_m g) = (J_n f \;\mid\; J_m g)_{L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})} = 0$$

for all $f \in L^2(S_n(T)$ and $g \in L^2(S_m(T))$.

Proof: J_n is the composition of two linear isometries, namely

$$L^2(S_n(T)) \overset{1}{\hookrightarrow} L_T^2(W) \overset{\cong}{\to} L^2(\Omega, \mathcal{F}_T, \mathbb{P}), \;\; f \mapsto Y^{(n)} f \mapsto (Y^{(n)} f \bullet W)_T$$ □

Until now, we have defined the multiple Wiener integral only for functions belonging to $L^2(S_n(T))$ with given $T > 0$. The definition of the simplex $S_n(T) = \{(t_1, t_2, ..., t_n) \in [0, T]^n : t_1 \leq t_2 \leq ... \leq t_n\}$ leads to the conclusion that the order structure of $\mathbb{R}^n$ plays a fundamental role in the construction of the mutiple integral, but it is not really so. Let $f \in L^2([0, T]^n)$ be a *symmetric* function; i.e.: $f(t_1, t_2, ..., t_n) = f(t_{\sigma(1)}, t_{\sigma(2)}, ..., t_{\sigma(n)})$ for all permutations $\sigma \in \mathfrak{S}_n$. Let $\widehat{L^2}([0, T]^n)$ denote the (closed) linear subspace of all symmetric functions in $L^2([0, T]^n)$.

Proposition 2.2 *Let $n \in \mathbb{N}$ and $T > 0$. Then*[3]

$$\|f\|_{L^2([0,T]^n)}^2 = n! \cdot \|f \mid S_n(T)\|_{L^2(S_n(T))}^2 \tag{3}$$

for all $f \in \widehat{L^2}([0, T]^n)$.

[3]Often, we will denote $f \mid S_n(T)$ by the same symbol f, if no misunderstanding is possible.

PROOF: Nothing is to show for $n = 1$. So, let (3) be true for $n \in \mathbb{N}$ and let $f \in \widehat{L^2}([0,T]^{n+1})$. Set $H_k = L^2([0,T]^k)$. Then, by induction assumption and Fubini

$$\begin{aligned}\|f\|^2_{H_{n+1}} &= \int_0^T \|f(\cdot,s)\|^2_{H_n}\, ds = n! \int_0^T \|f(\cdot,s)\|^2_{S_n(T)}\, ds \\ &= n! \int_{S_n(T)\times[0,T]} f^2(t)\, d^{n+1}t\ .\end{aligned}$$

Now,

$$S_n(T)\times[0,T] = \{(\tau,s)\in[0,T]^{n+1} : \tau\in S_n(T)\} = A_{n+1}(T)\cup S_{n+1}(T)\ ,$$

where $A_{n+1}(T) := \{(\tau,s)\in[0,T]^{n+1} : \tau\in S_n(T),\ s<\tau_n\}$. Since exactly n ordered positions for such an s are possible, $A_{n+1}(T)$ equals to the disjoint junion $\bigcup_{k=1}^n \Lambda_k^{-1}(S_{n+1}(T))$ (a.s.), where $\Lambda_k \in O(n+1;\mathbb{R})$ denotes the matrix which maps the unit vector e_l to e_l if $1\le l<k$, e_l to e_{l+1} if $k\le l<n+1$ and e_{n+1} to e_k. Since f is symmetric, it follows in particular that $f^2(t) = f^2(\Lambda_k t)$ for all $k\in\{1,...,n\}$ and $t\in[0,T]^{n+1}$, so that the transformation formula implies that

$$\int_{S_{n+1}(T)} f^2(t)\, d^{n+1}t\ = \int_{\Lambda_k^{-1}(S_{n+1}(T))} f^2(\Lambda_k t)\, d^{n+1}t = \int_{\Lambda_k^{-1}(S_{n+1}(T))} f^2(t)\, d^{n+1}t,$$

and the proof is finished. □

Using this fact, we now transfer the definition of the multiple Wiener integral to functions $f\in\widehat{L^2}([0,T]^n)$ in the following sense:

$$I_n f := n!\cdot J_n(f\mid S_n(T)).$$

By the previous considerations, $I_n : \widehat{L^2}([0,T]^n)\to L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ is a linear continuous operator, and:

$$\mathbb{E}((I_n f)^2) = \|I_n f\|^2_{L^2(\Omega,\mathcal{F}_T,\mathbb{P})} = n!\cdot\|f\|^2_{L^2([0,T]^n)}.$$

3 Hermite Polynomials and Chaos Decomposition

In the following paragraph we consider functions in $\widehat{L^2}([0,T]^n)$ of type

$$g^{\otimes n}(x_1,...,x_n) := \prod_{i=1}^n g(x_i)\ ,$$

where $g\in L^2([0,T])$. We will recognize that these *symmetric products* are the cornerstones of the factorization of $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ through multiple Wiener integrals. To that end, let us recall the well known *Hermite polynomials* ($n\in\mathbb{N}_0, x\in\mathbb{R}$) :

$$h_n(x) := (-1)^n\cdot\exp(\frac{x^2}{2})\cdot g^{(n)}(x)\ ,$$

where $g(x) := \exp(-\frac{x^2}{2})$. The first Hermite polynomials are $h_0(x)=1$, $h_1(x)=x$, $h_2(x)=x^2-1$ and $h_3(x)=x^3-3x$. Given $a>0$, we put

$$H_n(x,a) := \sqrt{a^n}\cdot h_n(\frac{x}{\sqrt{a}}).$$

Then we have the following

Lemma 3.1 *Let $t \in \mathbb{R}, x \in \mathbb{R}$ and $a > 0$. Then*

$$(i) \qquad \exp(tx - \tfrac{t^2}{2}) = \sum_{n=0}^{\infty} \tfrac{t^n}{n!} \cdot h_n(x)$$

$$(ii) \qquad \exp(tx - \tfrac{at^2}{2}) = \sum_{n=0}^{\infty} \tfrac{t^n}{n!} \cdot H_n(x,a).$$

PROOF: Let $x \in \mathbb{R}$ fixed. Since $\exp(tx - \frac{t^2}{2}) = \exp(\frac{x^2}{2}) \cdot (g \circ \tau_x)(t)$, where $\tau_x(t) := x - t$ and $g(y) := \exp(-\frac{y^2}{2})$, Taylor's formula applied to $g \circ \tau_x$ leads to

$$\begin{aligned}
\exp(tx - \frac{t^2}{2}) &= \exp(\frac{x^2}{2}) \cdot \sum_{n=0}^{\infty} \frac{(g \circ \tau_x)^{(n)}(0)}{n!} \cdot t^n \\
&= \exp(\frac{x^2}{2}) \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(g^{(n)} \circ \tau_x)(0)}{n!} \cdot t^n \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot h_n(x) \ ,
\end{aligned}$$

and (i) is proven. Now, (ii) follows directly, by setting $\exp(tx - \frac{at^2}{2}) = \exp(sy - \frac{s^2}{2})$, where $s := t\sqrt{a}$ and $y := \frac{x}{\sqrt{a}}$. □

Due to partial differentiation of the function $(x,a) \mapsto \exp(tx - \frac{at^2}{2})$, the corollary immediately implies the important

Remark 3.1 $\frac{\partial}{\partial x} H_n(x,a) = n \cdot H_{n-1}(x,a)$ *and* $(\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial a}) H_n(x,a) = 0$ *on* $\mathbb{R} \times \mathbb{R}_+^*$.

Now we can prove the following representation:

Theorem 3.1 *Let $T > 0$ and $g \in L^2([0,T])$. Then $g^{\otimes n} \in \widehat{L^2}([0,T]^n)$ for all $n \in \mathbb{N}$ and*

$$I_n(g^{\otimes n}) = H_n(X_T \ , \ \langle X, X \rangle_T) = H_n((g \bullet W)_T \ , \ \|g\|^2_{L^2([0,T])}) \ , \tag{4}$$

where $X := g \bullet W$.

PROOF: We will prove the statement by induction on n. Nothing is to show for $n = 1$. Let (4) be valid for $n \in \mathbb{N}$ and set $\phi_{n+1} := g^{\otimes n+1} \mid S_{n+1}(T)$. Then $\phi_{n+1}(\cdot, s) = (g \mid [0,s])^{\otimes n} \cdot g(s)$ on $S_n(s)$ for all $s \in [0,T]$, and the definition of I_{n+1} implies that

$$\begin{aligned}
I_{n+1} g^{\otimes n+1} &= (n+1)! \cdot J_{n+1}\phi_{n+1} = (n+1)! \cdot \int_0^T (Y^{(n+1)}\phi_{n+1})_s \, dW_s \\
&= (n+1)! \cdot \int_0^T (Y^{(n)}\phi_{n+1}(\cdot, s) \bullet W)_s \, dW_s \\
&= (n+1)! \cdot \int_0^T g(s) \cdot (Y^{(n)}(g \mid [0,s])^{\otimes n} \bullet W)_s \, dW_s \\
&= (n+1) \cdot \int_0^T g(s) \cdot I_n(g \mid [0,s])^{\otimes n} \ dW_s \\
&= (n+1) \cdot \int_0^T g(s) \cdot H_n(X_s \ , \ \langle X, X \rangle_s) \ dW_s \ .
\end{aligned}$$

On the other hand, using the previous remark, *Itô's formula* applied to $H_{n+1}(X_T\ ,\ \langle X,X\rangle_T)$ leads to

$$\begin{aligned} H_{n+1}(X_T\ ,\ \langle X,X\rangle_T) &= \int_0^T D_1 H_{n+1}(X_s\ ,\ \langle X,X\rangle_s)\, dX_s + 0 \\ &= (n+1)\cdot \int_0^T H_n(X_s\ ,\ \langle X,X\rangle_s)\, dX_s \\ &= (n+1)\cdot \int_0^T H_n(X_s\ ,\ \langle X,X\rangle_s)\cdot g(s)\, dW_s\ , \end{aligned}$$

and the proof is finished. □

Let $T > 0$. To prove our main theorem in this paragraph - the Chaos representation of L^2 -, we need a deeper investigation of the set $\{\exp((g\bullet W))_T : g \in L^2([0,T])\}$. Let $g \in L^2([0,T])$. Then $X := g \bullet W \in M_T^2(W)$ and $\exp(X_T) = \mathcal{E}(X)_T \cdot \exp(\frac{1}{2}\|g\|^2_{L^2([0,T])})$ is an element of $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ with $\mathbb{E}(\exp(X_T)) \le \exp(\frac{1}{2}\|g\|^2_{L^2([0,T]}) < \infty$ (since $\mathcal{E}(X)$ is a positive supermartingale[4]), but in our case, we are able to show the following:

Lemma 3.2

(*i*) $\exp((g\bullet W)_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *for all* $g \in L^2([0,T])$;

(*ii*) $\{\exp((g\bullet W)_T) : g \in L^2([0,T])\}$ *is total in* $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Proof: ad (i): Let $X = g \bullet W$. Due to Lemma 5 and the previous theorem, it follows (for $t = 1$) that

$$\mathcal{E}(X)_T = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot H_n(X_T, < X, X >_T) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g^{\otimes n})$$

holds pointwise on Ω. Now we show that this convergence still holds in $L^2(\Omega)$. Let us consider the finite sums $Z_n := \sum_{k=0}^{n} \frac{1}{k!} I_k(g^{\otimes k}) = \sum_{k=0}^{n} J_k(g^{\otimes k} \mid S_k(T))$ and let $m < n$. Each Z_n belongs to $L^2(\Omega)$, and it follows (by orthogonality and isometry) that $\|Z_n - Z_m\|^2_{L^2(\Omega)} = \sum_{k=m+1}^{n} \frac{1}{k!}\|g^{\otimes k}\|^2_{L^2(S_k(T))} = \sum_{k=m+1}^{n} \frac{1}{k!}(\|g\|^2_{L^2([0,T])})^k$. Hence (Z_n) is a Cauchy sequence in $L^2(\Omega)$, and it follows the existence of a $Z \in L^2(\Omega)$ so that $\|Z_n - Z\|_{L^2(\Omega)} \to 0$ for $n \to \infty$. In particular, there exists a subsequence (Z_{n_k}) of (Z_n) so that $Z = \lim_{k\to\infty} Z_{n_k}$ pointwise in Ω. Thus $\mathcal{E}(X)_T = Z \in L^2(\Omega)$ and therefore $\exp(X_T) = Z \cdot \exp(\frac{1}{2}\|g\|^2_{L^2([0,T]}) \in L^2(\Omega)$.

ad (ii): Let N be the L^2-closure of the linear hull M of the set $\{\exp((g\bullet W)_T) : g \in L^2([0,T])\}$. Since $L^2(\Omega) = N \oplus N^\perp$ and $N^\perp = M^\perp$, we only have to show that $M^\perp = 0$. So let $\xi \in M^\perp$. By linearity of the stochastic integral, it follows that

$$\mathbb{E}(\xi \cdot \exp(\sum_{i=1}^{n} \lambda_i \cdot (g_i \bullet W)_T)) = 0$$

[4]Indeed, Novikov's condition implies that $\mathcal{E}(X)$ is still an uniformly integrable *martingale*.

for all $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $g_1, ..., g_n \in L^2([0,T])$. In particular we obtain for arbitrary $t_1, ..., t_n \in [0,T]$

$$\mathbb{E}(\xi \cdot \exp((\lambda \mid Z(\omega))_{\mathbb{R}^n})) = 0 \ , \tag{5}$$

where $\lambda := (\ \lambda_1, ..., \lambda_n)$ and $Z := ((\mathbf{1}_{(0,t_1]} \bullet W)_T, ...(\mathbf{1}_{(t_{n-1},t_n]} \bullet W)_T) = (W_{t_1}, ..., W_{t_n} - W_{t_{n-1}})$. By construction of Z, $w \mapsto \mathbb{E}(\xi \cdot \exp((w \mid Z)_{\mathbb{R}^n}))$ is an entire function on the connected open domain $\mathbb{C}^n$, so that (5) is true for all $\lambda \in \mathbb{C}^n$. Denoting $m(A) := \int_A \xi \ d\mathbb{P} \ \ (A \in \mathcal{F}_T^W)$, it follows in particular that

$$\begin{aligned} 0 &= \mathbb{E}(\xi \cdot \exp(i \ (\lambda \mid Z(\omega))_{\mathbb{R}^n})) = \int_\Omega \exp(i \ (\lambda \mid Z(\omega))_{\mathbb{R}^n}) \ m(d\omega) \\ &= \int_{\mathbb{R}^n} \exp(i \ (\lambda \mid \rho)_{\mathbb{R}^n}) \ \ m_Z(d\rho) = \widehat{m_Z}(\lambda) \end{aligned}$$

for all $\lambda \in \mathbb{R}^n$. In other words, the Fourier transform of the measure m_Z vanishes on $\mathbb{R}^n$. Hence, by injectivity of the Fourier transform, m vanishes on $\sigma(Z) = \sigma(W_{t_1}, ..., W_{t_n})$ for all $t_1, ..., t_n \in [0,T]$, which implies by definition of m, that $\mathbb{E}(\mathbf{1}_A \cdot \xi) = m(A) = 0$ for all $A \in \mathcal{F}_T^W$. Hence, $\xi = 0$, and the proof of the lemma is finished. □

Now we are totally prepared to prove

Theorem 3.2 (Chaos decomposition) *Let $T > 0$ and $X \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$. Then there exists a unique sequence $(f_n)_{n\in\mathbb{N}}$ of deterministic and symmetric functions $f_n \in \widehat{L^2}([0,T]^n)$ so that*

$$X = \mathbb{E}(X) + \sum_{n=1}^{\infty} I_n f_n$$

and

$$\|X\|^2_{L^2(\Omega)} = \mathbb{E}^2(X) + \sum_{n=1}^{\infty} n! \cdot \|f_n\|^2_{L^2([0,T])} \ .$$

Proof: Let $X \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ be given. Then, by the previous lemma, there exists a sequence (Z_n) belonging to the linear hull of the set $\{\exp((g \bullet W)_T) : g \in L^2([0,T])\}$ so that $\|X - Z_n\|^2_{L^2(\Omega)} \to 0$. Each Z_n can be written as a finite sum of type $\sum_{k=1}^{l_n} \alpha_k \exp((g_k \bullet W)_T)$ with real α_k and square-integrable g_k. By theorem 6 and the previous considerations, each stochastic exponential $\mathcal{E}(g_k \bullet W)_T$ can be written as $\mathcal{E}(g_k \bullet W)_T = \sum\limits_{m=0}^{\infty} \frac{1}{m!} I_m(g_k^{\otimes m})$, so that $Z_n = \sum_{m=0}^{\infty} J_m \phi_m^n$ with $(\phi_m^n)_{n\in\mathbb{N}} \subseteq L^2(S_m(T))$. Orthogonality and the isometry condition now lead to $\|Z_i - Z_j\|^2_{L^2(\Omega)} = \sum_{m=0}^{\infty} \|\phi_m^i - \phi_m^j\|^2_{L^2(S_m(T))}$ for all $i, \ j \in \mathbb{N}$. Thus, $(\phi_m^i)_{i\in\mathbb{N}}$ is a $L^2(S_m(T))$-Cauchy sequence for every $m \in \mathbb{N}_0$, implying the existence of a limit $\phi_m \in L^2(S_m(T))$ with $\|\phi_m - \phi_m^i\|^2_{L^2(S_m(T))} \to 0$ for $i \to \infty$, and we obtain that $\sum_{m=0}^{\infty} \|\phi_m - \phi_m^i\|^2_{L^2(S_m(T))} \to 0$ for $i \to \infty$. Hence, by orthogonality and the isometry condition again, it follows the existence of $Z := \sum_{m=0}^{\infty} J_m \phi_m = \sum_{m=0}^{\infty} J_m(\phi_m - \phi_m^{i_0}) + Z_{i_0} \in L^2(\Omega)$ (for a suitable $i_0 \in \mathbb{N}$), and we obtain that $\|Z - Z_i\|^2_{L^2(\Omega)} = \sum_{m=0}^{\infty} \|\phi_m - \phi_m^i\|^2_{L^2(S_m(T))} \to 0$ for $i \to \infty$. By uniqueness of the limits, $X = Z = \sum_{m=0}^{\infty} J_m \phi_m$ with $(\phi_m) \subseteq L^2(S_m(T))$.

To finish our proof we first extend each ϕ_m trivially to $\psi_m \in L^2([0,T]^m)$ and consider then the symmetrization $\widetilde{\psi_m}$ of ψ_m:

$$\widetilde{\psi_m} := \frac{1}{m!} \cdot \sum_{\sigma \in \mathfrak{S}_n} \psi_m \circ A_\sigma \in \widehat{L^2}([0,T]^m) \ ,$$

where $A_\sigma(t_1,...,t_m) := (t_{\sigma(1)},...,t_{\sigma(m)})$ for all $(t_1,...,t_m) \in [0,T]^m$. Since $A_\sigma(S_m(T))$ has no common points with $S_m(T)$ for all $\sigma \neq id$, the definition of ψ_m implies that $(\psi_m \circ A_\sigma) \mid S_m(T) = 0$ for all $\sigma \neq id$, so that $\widetilde{\psi_m} \mid S_m(T) = \frac{1}{m!} \cdot \phi_m$, and we obtain that $X = \sum_{m=0}^{\infty} J_m \phi_m = \sum_{m=0}^{\infty} I_m \widetilde{\psi_m}$. Moreover, due to proposition 4, it follows that

$$\|X\|^2_{L^2(\Omega)} = \sum_{m=0}^{\infty} \|\phi_m\|^2_{L^2(S_m(T))} = \sum_{m=0}^{\infty} m! \cdot \|\widetilde{\psi_m}\|^2_{L^2([0,T]^m)} \ .$$

Since $X = \sum_{m=0}^{\infty} J_m \phi_m = \phi^0 + \sum_{m=1}^{\infty} (Y^{(m)} \phi_m \bullet W)_T$, we have $\mathbb{E}(X) = \phi^0 + 0$, and because of the orthogonal representation, uniqueness follows immediately. □

References

[1] H. P. Bermin. *Path Dependent Options: Hedging Lookback and Partial Lookback Options*, Preprint, Lund University Sweden.

[2] J. M. Bismut. *Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions,* Z. Wahrsch. Verw. Gebiete **56** (1981), 469-505.

[3] M. Dritschel and P. Protter. *Complete Markets with Discontinuous Security Price*, Preprint, Dept. Statistics, Purdue University, West Lafayette, USA.

[4] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. *An Application of Malliavin Calculus to Monte Carlo Methods in Finance*, Forthcoming in Finance & Stochastics.

[5] I. Karatzas, D. Ocone, and J. Li. *An extension of Clark's formula,* Stochastics and Stochastic Reports, **37** (1991), 127-131.

[6] N. Kazamaki. *Continuous Exponential Martingales and BMO,* Springer, Lecture Notes in Mathematics 1579 (1994).

[7] H. Kunita and S. Watanabe. *On square integrable martingales*, Nagoya Math. J. **30** (1967), 209-245.

[8] V. Lacoste. *Wiener Chaos: A New Approach to Option Hedging*, Math. Finance, Vol **6**, No. 2 (April 1996), 197–213.

[9] P. Malliavin. *Stochastic calculus of variations and hypoelliptic operators*, Proceedings of the International Conference on Stochastic Differential Equations, Kyoto, 195–263, Kinokuniya, Tokyo; Wiley, New York (1976)

[10] P. A. Meyer. *Quantum Probability for Probabilists*, Springer, Lecture Notes in Mathematics 1538, 2nd ed. (1994).

[11] D. Nualart. *The Malliavin Calculus and Related Topics*, Springer, Probability and its Applications (1995).

[12] B. Øksendal. *An introduction to Malliavin Calculus with Applications to Economics*, Norwegian School of Economics and Business Administration, Working Paper no. 3/96 (1996).

[13] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, Springer, Grundlehren der math. Wissensch., 2nd ed. (1994).

[14] D. Stroock. *The Malliavin calculus and its applications*, Stochastic Integrals (Proc. Sympos. Univ. Durham, Durham 1980), 394-432, Springer, Lecture Notes in Mathematics 851 (1981).