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Operator Ideals and the Principle of Local Reflexivity

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0.1. Introduction

Our aim is, to give necessary and sufficient conditions which allow us to transform the local reflexivity principle of Lindenstrauss and Rosenthal [Li–Rt] from the canonical operator norm $\|\cdot\|$ to p -Banach ideal norms $\|\cdot\|_{\mathcal{A}}$, where $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a given p -Banach ideal ($0 < p \leq 1$).

We will recognize two important facts:

- By a natural generalization of the *weak \mathcal{A} -local reflexivity principle* (introduced in [Oe1] and [Oe2]), we can omit the assumed maximality of the p -Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ in theorem 2.9. of [Oe2]. Moreover we are allowed to consider all $0 < p \leq 1$ and not only the case $p = 1$.
- There are interesting relations between the above mentioned generalization of weak local reflexivity and structural properties of the ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ such as *accessibility* (introduced in [D] and [D–F]). Hence, tensor norms are involved (cf. [Oe2]).

0.2. Notation and terminology

We shall use the common notations of Banach-space-theory; in particular B_E denotes the closed unit ball of a normed space E (over $K = \mathbb{R}$ or \mathbb{C}), E' the dual space of E and $\mathcal{L}(E, F)$ is the class of all (continuous) operators between the normed spaces E and F . Given $T \in \mathcal{L}(E, F)$, the dual operator of T is denoted by T' . *NORM*, *BAN* and *FIN* denotes the class of all normed spaces, Banach spaces and finite dimensional spaces respectively. *FIN*(E) is the class of all finite dimensional subspaces of a normed space E and *COFIN*(E) is the class of all finite codimensional subspaces of E . Concerning operator ideals we follow Pietsch's book ([P]). If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ are both normed operator ideals, we sometimes use the abbreviation $\mathcal{A} = \mathcal{B}$ to indicate the equality $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) = (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and we write \mathcal{A}^d instead of \mathcal{A}^{dual} . If $T: E \rightarrow F$ is an operator, we indicate that it is a metric injection ($\|Tx\| =$

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= $\|x\|$) by writing

$$T: E \overset{1}{\hookrightarrow} F$$

and that it is a metric surjection (F has the quotient norm of E via T) by

$$T: E \overset{1}{\twoheadrightarrow} F.$$

If there exists an isometric isomorphism between the spaces E and F , we write $E \cong F$. For $G \in \text{FIN}(E)$, $J_G^E: G \overset{1}{\hookrightarrow} E$ denotes the canonical metric injection and for $G \in \text{COFIN}(E)$, G closed, $Q_G^E: E \overset{1}{\twoheadrightarrow} E/G$ denotes the canonical metric surjection. We assume the reader to be familiar with the basics of the general theory of tensor norms as they are presented in [Gr], [D] and [D-F]. Another important tool to describe local properties of ideal components is given by the *trace* on a normed space E which is the linearization of the duality bracket

$$\begin{aligned} E' \times E &\rightarrow \mathbb{K} \\ (a, x) &\mapsto \langle x, a \rangle, \end{aligned}$$

whence

$$\begin{aligned} \text{tr}: E' \otimes E &\rightarrow \mathbb{K} \\ \sum_{i=1}^n a_i \otimes x_i &\mapsto \sum_{i=1}^n \langle x_i, a_i \rangle. \end{aligned}$$

We recall that a Banach space E has the *metric approximation property* (short: m.a.p.) if for all compact sets $K \subseteq E$ and for all $\varepsilon > 0$ there is a finite dimensional operator $L \in \mathcal{F}(E, E)$ with $\|L\| \leq 1$ such that $\|Lx - x\| \leq \varepsilon$ for all $x \in K$. Finally we remember the important

Principle of local reflexivity: Let M and F be Banach spaces, M finite dimensional and $T \in \mathcal{L}(M, F)$. Then for every $\varepsilon > 0$ and $N \in \text{FIN}(F')$ there is an $R \in \mathcal{L}(M, F)$ such that

- (i) $\|R\| \leq (1 + \varepsilon) \|T\|$
- (ii) $\langle Rx, b \rangle = \langle b, Tx \rangle \forall (x, b) \in M \times N$
- (iii) $j_F Rx = Tx \forall x \in M$ with $Tx \in j_F(F)$.

1. The weak (\mathcal{A})-local reflexivity principle

In the following, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ always denotes a p -Banach ideal with $0 < p \leq 1$ fixed. Recall, that the *adjoint ideal* $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ is given by all operators $T \in \mathcal{L}(E, F)$ ($E, F \in \text{BAN}$) for which there exists a number $\varrho \geq 0$ such that for all $E_0, F_0 \in \text{BAN}$ and for all $A \in \mathcal{F}(F, F_0)$, $S \in \mathcal{A}(F_0, E_0)$, $B \in \mathcal{F}(E_0, E)$

$$|\text{tr}(TBSA)| \leq \varrho \cdot \|B\| \cdot \|S\|_{\mathcal{A}} \cdot \|A\|.$$

By definition, $\|T\|_{\mathcal{A}^*} := \inf(\varrho)$ where the infimum is formed by all such $\varrho \geq 0$ ([P]). According to [G-L-R] the *conjugate ideal* $(\mathcal{A}^{\Delta}, \|\cdot\|_{\mathcal{A}^{\Delta}})$ is given by all operators

$T \in \mathcal{L}(E, F)$ ($E, F \in BAN$) for which there exists a number $\varrho \geq 0$ such that for all $L \in \mathcal{F}(F, E)$

$$|tr(TL)| \leq \varrho \cdot \|L\|_{\mathcal{A}}.$$

By definition, $\|T\|_{\mathcal{A}^\Delta} := \inf(\varrho)$ where the infimum is formed by all such $\varrho \geq 0$.

1.1. Definition: Let $\varepsilon > 0$, F be a Banach space, $M \in FIN$ and $N \in FIN(F')$. We are talking about the *weak* (\mathcal{A}) -*local reflexivity principle* (short: (w) - (\mathcal{A}) -*l.r.p.*) if for every $T \in \mathcal{L}(M, F')$ there is an $S \in \mathcal{L}(M, F)$ such that

$$\langle b, Tx \rangle = \langle Sx, b \rangle \quad \forall (x, b) \in M \times N$$

and

$$\|S\|_{\mathcal{A}} \leq (1 + \varepsilon) \|T\|_{\mathcal{A}^{**}}.$$

Obviously the (w) - (\mathcal{A}) -*l.r.p.* always implies the (w) \mathcal{A} -*l.r.p.*, and if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a maximal Banach ideal ($p = 1$), then the (w) \mathcal{A} -*l.r.p.* implies the (w) - (\mathcal{A}) -*l.r.p.* To prove our main theorem 1.5., we need the following

1.1. Lemma: Let $L \in \mathcal{F}(E, F)$, $A \in \mathcal{L}(N, E')$ and $\varepsilon > 0$, where E, F are arbitrary Banach spaces and $\dim N < \infty$. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p -Banach ideal ($0 < p \leq 1$) such that the (w) - (\mathcal{A}) -*l.r.p.* holds. Then there is an operator $B \in \mathcal{L}(N, E)$ such that $\|B\|_{\mathcal{A}} \leq (1 + \varepsilon) \|A\|_{\mathcal{A}^{**}}$ and $L'A = L'j_E B = j_F LB$.

Proof: Since the range of L' is a finite dimensional subspace of E' , there is an operator $B \in \mathcal{L}(N, E)$ such that $\langle L'b, Ay \rangle = \langle By, L'b \rangle$ for all $b \in F'$, $y \in N$ and $\|B\|_{\mathcal{A}} \leq (1 + \varepsilon) \|A\|_{\mathcal{A}^{**}}$. Hence, for all $b \in F'$, $y \in N$ we have $\langle b, L'Ay \rangle = \langle LBy, b \rangle = \langle b, j_F LB y \rangle$. \square

Easy to prove, but nevertheless of importance is the following

1.1. Lemma: Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p -Banach ideal ($0 < p \leq 1$). Let $M \in FIN$ and $F \in BAN$. Then

$$\mathcal{A}^\Delta(F, M) \cong \mathcal{A}(M, F)'$$

where the isometric isomorphism is given by canonical trace duality. \square

Remember, that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called *left-accessible* if for all $(E, N) \in BAN \times FIN$, $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $L \in COFIN(E)$, $S \in \mathcal{L}(E/L, N)$ such that $T = SQ_L^E$ and $\|S\|_{\mathcal{A}} \leq (1 + \varepsilon) \|T\|_{\mathcal{A}} ([D], [D-F])$. By using tensor norm techniques (!) the following non-trivial result can be shown:

1.4. Proposition: Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach ideal and $E, F \in BAN$ such that E' or F has the m.a.p. Then

$$\mathcal{B}^{\min}(E, F) \underset{\mathcal{A}}{\hookrightarrow} (\mathcal{B}^{*\Delta})^{dd}(E, F).$$

In particular $((\mathcal{A}^\Delta)^{dd}, \|\cdot\|_{(\mathcal{A}^\Delta)^{dd}})$ is left-accessible for each maximal Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$.

Prof: cf. [Oe1] and [Oe2]. \square

Now we have all prepared to prove our main

1.5. Theorem: Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p -Banach ideal ($0 < p \leq 1$). TFAE:

- (1) $(\mathcal{A}^{\Delta}, \|\cdot\|_{\mathcal{A}^{\Delta}})$ is left-accessible
- (2) $\mathcal{A}^{**}(M, F'') \cong \mathcal{A}(M, F)'' \forall (F, M) \in BAN \times FIN$
- (3) The (w) - (\mathcal{A}) -l.r.p. holds.

Proof: (1) \Rightarrow (2): Let $(\mathcal{A}^{\Delta}, \|\cdot\|_{\mathcal{A}^{\Delta}})$ be left-accessible and $(F, M) \in BAN \times FIN$. By [D–F, 25.2] it follows that $\mathcal{F} \circ \mathcal{A}^{\Delta} = (\mathcal{A}^*)^{\min}$ and so 1.3. implies that $\mathcal{A}(M, F)' \cong (\mathcal{A}^*)^{\min}(F, M)$. Hence dualization yields ([D–F, 22.6.]) $\mathcal{A}(M, F)'' \cong (A^{**}) \cdot (M, F'')$.

(2) \Rightarrow (3): This implication follows by using Helly's lemma ([P]) and the canonical trace duality 1.3; namely, observe that by assumption (2)

$$\mathcal{A}(M, F)'' \xrightarrow{\cong} \mathcal{A}^{**}(M, F'')$$

$$\xi \mapsto T_{\xi}$$

is an isometric isomorphism, where $\langle b, T_{\xi}x \rangle := \langle \text{tr}((b \otimes x) \cdot), \xi \rangle$ ($x \in M, b \in F''$). Let $\varepsilon > 0$, $N \in FIN(F')$ and $T \in \mathcal{L}(M, F')$. Let $\{x_1, \dots, x_n\}$ be a basis of M and $\{b_1, \dots, b_m\}$ be a basis of $N \subseteq F'$. Let $L_{ij} := b_i \otimes x_j$. By 1.3., the linear span of $\{\text{tr}(L_{ij} \cdot) : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a finite dimensional subspace of $\mathcal{A}(M, F)'$. By assumption, there is a $\xi_0 \in \mathcal{A}(M, F)''$, such that $\langle b_i, Tx_j \rangle = \langle \text{tr}(L_{ij} \cdot), \xi_0 \rangle \forall i \in \{1, \dots, m\} \forall j \in \{1, \dots, n\}$. By Helly, there exists an $S \in \mathcal{A}(M, F)$ with $\|S\|_{\mathcal{A}} \leq (1 + \varepsilon) \|\xi_0\| = (1 + \varepsilon) \|T\|_{\mathcal{A}^{**}}$ and with

$$\langle \text{tr}(L_{ij} \cdot), \xi_0 \rangle = \langle S, \text{tr}(L_{ij} \cdot) \rangle = \text{tr}(L_{ij}S) = \langle Sx_j, b_i \rangle$$

for all i and j . Hence – by linearity of T – the claim follows.

(3) \Rightarrow (1): Let $\mathcal{B} := \mathcal{A}^{**}$. Since $((\mathcal{B}^{\Delta})^{dd}, \|\cdot\|_{(\mathcal{B}^{\Delta})^{dd}})$ is left-accessible (by 1.4.), it suffices to show that for all $(E, N) \in BAN \times FIN$ and for all $L \in \mathcal{L}(E, N)$ we have

$$(*) \quad \|L'\|_{\mathcal{B}^{\Delta}} = \|L\|_{\mathcal{A}^{\Delta}}.$$

Obviously, $\|L\|_{\mathcal{A}^{\Delta}} \leq \|L\|_{\mathcal{B}^{\Delta}} \leq \|L'\|_{\mathcal{B}^{\Delta}}$. To prove the other inequality we use lemma 1.2.. Let $A \in \mathcal{F}(N'', E'')$ be given. By assumption we can choose an operator $B \in \mathcal{L}(N'', E)$ as in lemma 1.2.. It follows that

$$|\text{tr}(L'A)| = |\text{tr}(j_F LB)| \leq \|L\|_{\mathcal{A}^{\Delta}} \|B\|_{\mathcal{A}} \leq (1 + \varepsilon) \|L\|_{\mathcal{A}^{\Delta}} \|A\|_{\mathcal{A}^{**}}.$$

Hence $\|L'\|_{\mathcal{B}^{\Delta}} \leq \|L\|_{\mathcal{A}^{\Delta}}$ and $(*)$ is proven. \square

Until now we do not know, if there exists a *maximal* Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ such that the (w) - (\mathcal{A}) -l.r.p. does not hold. In this case $(\mathcal{A}^{\Delta}, \|\cdot\|_{\mathcal{A}^{\Delta}})$ is not left-accessible, and especially $(\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$ would be another candidate for a non (left-) accessible maximal Banach ideal (since $(\mathcal{A}^{\Delta}, \|\cdot\|_{\mathcal{A}^{\Delta}}) \subseteq (\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})$). Indeed, the hard problem of constructing such a candidate was open for a long time and had been recently solved by Pisier on the Oberwolfach meeting in September 1991, using a factorization over his own Pisier space P (cf. [D–F, 31.6.]). Therefore it seems also very

non-trivial to construct a maximal Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, for which the (w) - (\mathcal{A}) - $l.r.p.$ does not hold.

1.5. Remark: There exists a *minimal* Banach ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ such that the (w) - (\mathcal{A}) - $l.r.p.$ does not hold.

Proof: Let $\mathcal{A} := \mathcal{A}_0^{\min}$, where \mathcal{A}_0^* is Pisier's counterexample of a non left-accessible, maximal Banach ideal. Since in general $(\mathcal{B}^{\min}, \|\cdot\|_{\mathcal{B}^{\min}}) \subseteq ((\mathcal{B}^*)^{\Delta}, \|\cdot\|_{(\mathcal{B}^*)^{\Delta}})$, it follows for arbitrary $L \in \mathcal{F}(E, F)$ ($E, F \in BAN$) that

$$\|L\|_{\mathcal{A}^*} \leq \|L\|_{(\mathcal{A}^{\min})^{\Delta}} \leq \|L\|_{(\mathcal{A}^*)^{\Delta\Delta}} \leq \|L\|_{\mathcal{A}^*}.$$

Therefore the left-accessibility of \mathcal{A}^{Δ} would imply the left-accessibility of \mathcal{A}_0^* , which is a contradiction. \square

1. The local reflexivity principle for operator ideals

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p -Banach ideal ($0 < p \leq 1$) such that the (w) - (\mathcal{A}) - $l.r.p.$ holds. Then it is possible to transfer the principle of local reflexivity in the following sense:

2.1. Theorem: Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p -Banach ideal ($0 < p \leq 1$) such that the (w) - (\mathcal{A}) - $l.r.p.$ holds. Let $\varepsilon > 0$, $M \in FIN$, $F \in BAN$, $N \in FIN(F')$ and $S \in \mathcal{L}(M, F'')$. Then there exists an operator $R \in \mathcal{L}(M, F)$ such that

- (i) $\|R\|_{\mathcal{A}} \leq (1 + \varepsilon) \|S\|_{\mathcal{A}^{**}}$
- (ii) $\langle Rx, b \rangle = \langle b, Sx \rangle \quad \forall (x, b) \in M \times N$
- (iii) $j_F Rx = Sx \quad \forall x \in M \quad \text{with} \quad Sx \in j_F(F)$.

Proof: Let $M_0 := \{x \in M : Sx \in j_F(F)\}$ and $J : M_0 \hookrightarrow M$ the canonical embedding. Let $S_0 : M_0 \rightarrow F$, $x \mapsto j_F^{-1}(Sx)$. Let $N \subseteq L \subseteq F'$ with $\dim L < \infty$ and $\varepsilon > 0$. By assumption there exists an $R_L \in \mathcal{F}(M, F)$ such that $\|R_L\|_{\mathcal{A}} \leq (1 + \varepsilon) \|S\|_{\mathcal{A}^{**}}$ and $\langle R_L x, b \rangle = \langle b, Sx \rangle$ for all $(x, b) \in M \times L$. Hence

$$(*) \quad \langle R_L Jx, b \rangle = \langle b, j_F S_0 x \rangle = \langle S_0 x, b \rangle \quad \forall (x, b) \in M_0 \times L.$$

Let $\Phi := \{L \in FIN(F') : N \subseteq L\}$. By canonical set inclusion, Φ is a partially ordered set. Let $A = \sum_{i=1}^n b_i \otimes x_i \in \mathcal{L}(F, M_0)$ be arbitrary given $(b_1, \dots, b_n \in F'$ and $x_1, \dots, x_n \in M_0)$. Choose $L_0 \in \Phi$ such that $b_1, \dots, b_n \in L_0$. Hence, by (*) we obtain for all $L \in \Phi$ with $L \supseteq L_0$:

$$\text{tr}(R_L J A) = \sum_{i=1}^n \langle R_L J x_i, b_i \rangle = \sum_{i=1}^n \langle S_0 x_i, b_i \rangle = \text{tr}(S_0 A).$$

By the canonical trace duality 1.3., it follows that S_0 is the $\sigma(\mathcal{A}(M_0, F), \mathcal{A}(M_0, F)')$ -limit of the net $(R_L J)_{L \in \Phi}$. Now we consider the set C , consisting of all operators UJ where $U \in \mathcal{L}(M, F)$, $\|U\|_{\mathcal{A}} \leq (1 + \varepsilon) \|S\|_{\mathcal{A}^{**}}$ and $\langle Ux, b \rangle = \langle b, Sx \rangle$ for all $(x, b) \in M \times N$. Since $R_L J \in C$ for all $L \in \Phi$, S_0 is an element of the $\sigma(\mathcal{A}(M_0, F)$,

$\mathcal{A}(M_0, F)'$ -closure of the convex set C , hence S_0 is an element of the $\|\cdot\|_{\mathcal{A}}$ -closure of C . Therefore to each $\delta > 0$ there exists an $U_0 J \in C$ such that $\|S_0 - U_0 J\|_{\mathcal{A}} < \delta$. Let $Q : M \rightarrow M_0$ an arbitrary projection. Then $\|Q\| \leq 1$ and evidently the statements (ii) and (iii) are valid for the operator $R := (S_0 - U_0 J) Q + U_0 \in \mathcal{L}(M, F)$. Since

$$\|R\|_{\mathcal{A}} \leq \|S_0 - U_0 J\|_{\mathcal{A}} + \|U_0\|_{\mathcal{A}} < \delta + (1 + \varepsilon) \|S\|_{\mathcal{A}^{**}},$$

statement (i) follows, and the theorem is proven. \square

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