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Extension of Finite Rank Operators and Operator Ideals with the Property (I)

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Abstract. We present criteria and related techniques which help to decide whether the adjoint operator ideal $(\mathfrak{A}^*, \mathbf{A}^*)$ of an injective and totally accessible maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ is itself also totally accessible. This approach (which involves the transfer of the principle of local reflexivity to operator ideals) is based on the extension of finite rank operators, viewed as elements of the adjoint ideal $(\mathfrak{A}^*, \mathbf{A}^*)$. Using the local properties (I) and (S) of the corresponding product ideal $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, these methods even enable us to show that $\mathfrak{L}_\infty$ and $\mathfrak{L}_1$ cannot be totally accessible — answering an open question of Defant and Floret.

1. Introduction

In order to investigate the adjoint $(\mathfrak{A}^*, \mathbf{A}^*)$ of an injective and maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ concerning total accessibility, we study the behaviour of finite rank operators, viewed as elements of $(\mathfrak{A}^*, \mathbf{A}^*)$. In doing so, we assume that $(\mathfrak{A}^*, \mathbf{A}^*)$ allows a transfer of the norm estimation in the classical principle of local reflexivity to its ideal norm $\mathbf{A}^*$. Due to the local nature of this *principle of local reflexivity for operator ideals* (called $\mathfrak{A}$–LRP) — which had been introduced and discussed in [14] and [15] — and the local nature of maximal Banach ideals, local versions of injectivity (right–accessibility) resp. surjectivity (left–accessibility) of suitable operator ideals and factorizations through operators with finite dimensional range even imply interesting relations between operators with infinite dimensional range. Extending finite rank operators suitably, the $\mathfrak{A}^*$–LRP and the calculation of conjugate ideal norms then even give us sufficient conditions on $\mathfrak{A}^*$ to guarantee that each finite rank operator L has a finite rank–extension $\widetilde{L}$ so that $\mathbf{A}^*\big(\widetilde{L}\big) \le (1+\epsilon)\cdot\mathbf{A}^*(L)$ — for given $\epsilon > 0$. Consequently, we are lead to the problem under which circumstances a finite rank operator

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$L \in \mathfrak{A} \circ \mathfrak{B}$ has a factorization $L = AB$ so that $\mathbf{A}(A) \cdot \mathbf{B}(B) \leq (1+\epsilon) \cdot \mathbf{A} \circ \mathbf{B}(L)$ and A resp. B has finite dimensional range. Operator ideals $\mathfrak{A} \circ \mathfrak{B}$ with such a *property* (I) resp. *property* (S) had been introduced in [11] to prepare a detailed investigation of trace ideals.

After introducing the necessary framework, we recall the definition of the $\mathfrak{A}$–LRP and note some of its implications. We will see that the property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ plays a fundamental part in this paper. It not only leads us to an interesting result concerning operators acting between Banach spaces with cotype 2 (see Theorem 3.11). It even enables us to show that $\mathfrak{L}_\infty$ is not totally accessible — answering a question of Defant and Floret (see Theorem 3.15).

2. The framework

In this section, we recall the basic notation and terminology which we will use throughout in this paper. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard. We refer the reader to the monographs [3], [4] and [18] for the necessary background in operator ideal theory and the related terminology. Infinite dimensional Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are denoted throughout by W, X, Y and Z in contrast to the letters E, F and G which are used for finite dimensional Banach spaces only. The space of all operators (continuous linear maps) from X to Y is denoted by $\mathfrak{L}(X,Y)$, and for the identity operator on X, we write Id_X. The collection of all finite rank (resp. approximable) operators from X to Y is denoted by $\mathfrak{F}(X,Y)$ $\big($resp. $\overline{\mathfrak{F}}(X,Y)\big)$, and $\mathfrak{E}(X,Y)$ indicates the collection of all operators, acting between finite dimensional Banach spaces X and Y (elementary operators). The dual of a Banach space X is denoted by X', and X'' denotes its bidual $(X')'$. If $T \in \mathfrak{L}(X,Y)$ is an operator, we indicate that it is a metric injection by writing $T : X \overset{1}{\hookrightarrow} Y$, and if it is a metric surjection, we write $T : X \overset{1}{\twoheadrightarrow} Y$. If X is a Banach space, E a finite dimensional subspace of X and K a finite codimensional subspace of X, then $B_X := \{x \in X \mid \|x\| \leq 1\}$ denotes the closed unit ball, $J_E^X : E \overset{1}{\hookrightarrow} X$ the canonical metric injection and $Q_K^X : X \overset{1}{\twoheadrightarrow} X \diagup K$ the canonical metric surjection. Finally, $T' \in \mathfrak{L}(Y', X')$ denotes the dual operator of $T \in \mathfrak{L}(X,Y)$.

If $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ are given quasi–Banach ideals, we will use throughout the shorter notation $\big(\mathfrak{A}^{\mathrm{d}}, \mathbf{A}^{\mathrm{d}}\big)$ for the dual ideal and the abbreviation $\mathfrak{A} \overset{1}{=} \mathfrak{B}$ for the isometric equality $(\mathfrak{A}, \mathbf{A}) = (\mathfrak{B}, \mathbf{B})$. We write $\mathfrak{A} \subseteq \mathfrak{B}$ if, regardless of the Banach spaces X and Y, we have $\mathfrak{A}(X,Y) \subseteq \mathfrak{B}(X,Y)$. If X_0 is a fixed Banach space, we write $\mathfrak{A}(X_0, \cdot\,) \subseteq \mathfrak{B}(X_0, \cdot\,)$ (resp. $\mathfrak{A}(\,\cdot\,, X_0) \subseteq \mathfrak{B}(\,\cdot\,, X_0)$) if, regardless of the Banach space Z we have $\mathfrak{A}(X_0, Z) \subseteq \mathfrak{B}(X_0, Z)$ (resp. $\mathfrak{A}(Z, X_0) \subseteq \mathfrak{B}(Z, X_0)$). The metric inclusion $(\mathfrak{A}, \mathbf{A}) \subseteq (\mathfrak{B}, \mathbf{B})$ is often shortened by $\mathfrak{A} \overset{1}{\subseteq} \mathfrak{B}$. If $\mathbf{B}(T) \leq \mathbf{A}(T)$ for all finite rank (resp. elementary) operators $T \in \mathfrak{F}$ (resp. $T \in \mathfrak{E}$), we sometimes use the abbreviation $\mathfrak{A} \overset{\mathfrak{F}}{\subseteq} \mathfrak{B}$ $\Big($resp. $\mathfrak{A} \overset{\mathfrak{E}}{\subseteq} \mathfrak{B}\Big)$.

First we recall the basic notions of Grothendieck's metric theory of tensor products (cf., e. g., [3], [6], [8], [12]), which together with PIETSCH's theory of operator ideals spans the mathematical frame of this paper. A *tensor norm* α is a mapping which assigns to each pair (X, Y) of Banach spaces a norm $\alpha(\,\cdot\,; X, Y)$ on the algebraic tensor product $X \otimes Y$ $\big($shorthand: $X \otimes_\alpha Y$ and $X \widetilde{\otimes}_\alpha Y$ for the completion$\big)$ so that

a) $\varepsilon \le \alpha \le \pi$,

b) α satisfies the metric mapping property: If $S \in \mathfrak{L}(X, Z)$ and $T \in \mathfrak{L}(Y, W)$, then $\big\|S \otimes T : X \otimes_\alpha Y \longrightarrow Z \otimes_\alpha W\big\| \le \|S\| \, \|T\|$.

Well–known examples are the injective tensor norm ε, which is the smallest one, and the projective tensor norm π, which is the largest one. For other important examples we refer to [3], [6], or [12]. Each tensor norm α can be extended in two natural ways. For this, denote for given Banach spaces X and Y

$$\mathrm{FIN}(X) \ := \ \{E \subseteq X \mid E \in \mathrm{FIN}\} \quad \text{and} \quad \mathrm{COFIN}(X) \ := \ \{L \subseteq X \mid X/L \in \mathrm{FIN}\}\,,$$

where FIN stands for the class of all finite dimensional Banach spaces. Let $z \in X \otimes Y$. Then the *finite hull* $\overrightarrow{\alpha}$ is given by

$$\overrightarrow{\alpha}\,(z; X, Y) \ := \ \inf\{\alpha(z; E, F) \mid E \in \mathrm{FIN}(X),\ F \in \mathrm{FIN}(Y),\ z \in E \otimes F\}\,,$$

and the *cofinite hull* $\overleftarrow{\alpha}$ of α is given by

$$\overleftarrow{\alpha}\,(z; X, Y) := \sup\big\{\alpha\big(Q_K^X \otimes Q_L^Y(z); X/K, Y/L\big) \mid K \in \mathrm{COFIN}(X),\ L \in \mathrm{COFIN}(Y)\big\}\,.$$

α is called *finitely generated* if $\alpha = \overrightarrow{\alpha}$, *cofinitely generated* if $\alpha = \overleftarrow{\alpha}$ $\big($it is always true that $\overleftarrow{\alpha} \le \alpha \le \overrightarrow{\alpha}\big)$. α is called *right–accessible* if $\overleftarrow{\alpha}\,(z; E, Y) = \overrightarrow{\alpha}\,(z; E, Y)$ for all $(E, Y) \in \mathrm{FIN} \times \mathrm{BAN}$, *left–accessible* if $\overleftarrow{\alpha}\,(z; X, F) = \overrightarrow{\alpha}\,(z; X, F)$ for all $(X, F) \in \mathrm{BAN} \times \mathrm{FIN}$, and *accessible* if it is right–accessible and left–accessible. α is called *totally accessible* if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$. The injective norm ε is totally accessible, the projective norm π is accessible — but not totally accessible, and PISIER's construction implies the existence of a (finitely generated) tensor norm which is neither left– nor right–accessible (see [3], 31.6).

There exists a powerful one–to–one correspondence between finitely generated tensor norms and maximal Banach ideals which links thinking in terms of operators with "tensorial" thinking and which allows to transfer notions in the "tensor language" to the "operator language" and conversely. We refer the reader to [3] and [14] for detailed informations concerning this subject. Let X, Y be Banach spaces and $z = \sum_{i=1}^n x_i' \otimes y_i$ be an element in $X' \otimes Y$. Then $T_z(x) := \sum_{i=1}^n \langle x, x_i' \rangle y_i$ defines a finite rank operator $T_z \in \mathfrak{F}(X, Y)$ which is independent of the representation of z in $X' \otimes Y$. Let α be a finitely generated tensor norm and $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. α and $(\mathfrak{A}, \mathbf{A})$ are said to be *associated*, notation:

$$(\mathfrak{A}, \mathbf{A}) \ \sim \ \alpha \quad (\text{shorthand} : \mathfrak{A} \sim \alpha)\,,$$

if for all $E, F \in \mathrm{FIN}$

$$\mathfrak{A}(E, F) \ = \ E' \otimes_\alpha F$$

holds isometrically: $\mathbf{A}(T_z) = \alpha(z; E', F)$.

In addition to the maximal Banach ideal $(\mathfrak{L}, \|\cdot\|) \sim \varepsilon$ we mainly will be concerned with the maximal Banach ideals $(\mathfrak{I}, \mathbf{I}) \sim \pi$ (integral operators), $\big(\mathfrak{L}_2, \mathbf{L}_2\big) \sim w_2$ (Hilbertian operators), $\big(\mathfrak{D}_2, \mathbf{D}_2\big) \stackrel{1}{=} \big(\mathfrak{L}_2^*, \mathbf{L}_2^*\big) \stackrel{1}{=} \mathfrak{P}_2^{\mathrm{d}} \circ \mathfrak{P}_2 \sim w_2^*$ (2–dominated operators), $\big(\mathfrak{P}_p, \mathbf{P}_p\big) \sim g_p\backslash = g_q^*$ (absolutely p–summing operators), $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\big(\mathfrak{L}_\infty, \mathbf{L}_\infty\big) \stackrel{1}{=} \big(\mathfrak{P}_1^*, \mathbf{P}_1^*\big) \sim w_\infty$ and $\big(\mathfrak{L}_1, \mathbf{L}_1\big) \stackrel{1}{=} \big(\mathfrak{P}_1^{*\,\mathrm{d}}, \mathbf{P}_1^{*\,\mathrm{d}}\big) \sim w_1$. We also consider the maximal Banach ideals $\big(\mathfrak{C}_2, \mathbf{C}_2\big) \sim c_2$ (cotype 2 operators) and $\big(\mathfrak{A}_P, \mathbf{A}_P\big) \sim \alpha_P$ (Pisier's counterexample of a maximal Banach ideal which is neither right– nor left–accessible (cf. [3], 31.6)).

Since we will use them throughout in this paper, let us recall the important notions of the conjugate operator ideal (cf. [7], [11] and [15]) and the adjoint operator ideal (all details can be found in the standard references [3] and [18]). Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal.

1. Let $\mathfrak{A}^\Delta(X, Y)$ be the set of all $T \in \mathfrak{L}(X, Y)$ which satisfy

$$\mathbf{A}^\Delta(T) \ := \ \sup\{ |\mathrm{tr}(TL)| \mid L \in \mathfrak{F}(Y, X),\ \mathbf{A}(L) \leq 1\} \ < \ \infty \,.$$

Then a Banach ideal $\big(\mathfrak{A}^\Delta, \mathbf{A}^\Delta\big)$ is obtained (here, $\mathrm{tr}(\,\cdot\,)$ denotes the usual trace for finite rank operators). It is called the *conjugate ideal* of $(\mathfrak{A}, \mathbf{A})$.

2. Let $\mathfrak{A}^*(X, Y)$ be the set of all $T \in \mathfrak{L}(X, Y)$ which satisfy

$$\mathbf{A}^*(T) \ := \ \sup\big\{\big|\mathrm{tr}\big(TJ_E^X SQ_K^Y\big)\big| \mid E \in \mathrm{FIN}(X),\ K \in \mathrm{COFIN}(Y),\ \mathbf{A}(S) \leq 1\big\} \ < \ \infty \,.$$

Then a Banach ideal $(\mathfrak{A}^*, \mathbf{A}^*)$ is obtained. It is called the *adjoint operator ideal* of $(\mathfrak{A}, \mathbf{A})$.

By definition, it immediately follows that $\mathfrak{A}^\Delta \stackrel{1}{\subseteq} \mathfrak{A}^*$. Another easy, yet important observation is the following: let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal and $(\mathfrak{B}, \mathbf{B})$ be a quasi–Banach ideal. If $\mathfrak{A} \stackrel{\mathfrak{E}}{\subseteq} \mathfrak{B}$, then $\mathfrak{B}^* \stackrel{1}{\subseteq} \mathfrak{A}^*$, and $\mathfrak{A} \stackrel{\mathfrak{F}}{\subseteq} \mathfrak{B}$ implies the inclusion $\mathfrak{B}^\Delta \stackrel{1}{\subseteq} \mathfrak{A}^\Delta$. In particular, it follows that $\mathfrak{A}^{\Delta *} \stackrel{1}{=} \mathfrak{A}^{**}$ and $\big(\mathfrak{A}^{\Delta\Delta}\big)^* \stackrel{1}{=} \mathfrak{A}^*$.

Proposition 2.1. *Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal. If $\mathfrak{A} \stackrel{1}{=} \mathfrak{A}^{\mathrm{dd}}$, then $\mathfrak{A}^\Delta$ is regular.*

Proof. Let X, Y be arbitrary Banach spaces, $T \in \mathfrak{A}^{\Delta reg}(X, Y)$ and $L \in \mathfrak{F}(Y, X)$. Choose $A \in \mathfrak{F}(Y'', X)$ so that $L'' = j_X A$ $\big($if $L = T_z$ with $z = \sum_{i=1}^n y_i' \otimes x_i \in Y' \otimes X$, then $A = T_w$ where $w := \sum_{i=1}^n j_{Y'} y_i' \otimes x_i\big)$. Since $\mathfrak{A} \stackrel{1}{=} \mathfrak{A}^{\mathrm{dd}}$ in particular is regular, we have

$$|\mathrm{tr}(TL)| \ = \ |\mathrm{tr}(T'' j_X A)| \ = \ |\mathrm{tr}(j_Y T A)| \ \leq \ \mathbf{A}^\Delta(j_Y T) \cdot \mathbf{A}(A) \ = \ \mathbf{A}^{\Delta reg}(T) \cdot \mathbf{A}(L)\,,$$

and the claim follows. $\square$

Given quasi–Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$, let $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ be the corresponding product ideal and $\big(\mathfrak{A} \circ \mathfrak{B}^{-1}, \mathbf{A} \circ \mathbf{B}^{-1}\big)$ $\big($resp. $\big(\mathfrak{A}^{-1} \circ \mathfrak{B}, \mathbf{A}^{-1} \circ \mathbf{B}\big)\big)$ the corresponding "right–quotient" (resp. "left–quotient"). We write $\big(\mathfrak{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}}\big)$, to denote the *injective hull* of $\mathfrak{A}$, the unique smallest injective quasi–Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$, and $(\mathfrak{A}^{\mathrm{sur}}, \mathbf{A}^{\mathrm{sur}})$, the *surjective hull* of $\mathfrak{A}$, is the unique smallest surjective quasi–Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$. Of particular importance are the quotients $\mathfrak{A}^{\dashv} := \mathfrak{I} \circ \mathfrak{A}^{-1}$ and $\mathfrak{A}^{\vdash} := \mathfrak{A}^{-1} \circ \mathfrak{I}$ and their relations to $\mathfrak{A}^{\Delta}$ and $\mathfrak{A}^*$, treated in detail in [14] and [17]. Very useful will be the following statement which represents the injective hull (resp. the surjective hull) of a conjugate operator ideal as a certain quotient:

Proposition 2.2. *Let* $(\mathfrak{A}, \mathbf{A})$ *be an arbitrary quasi–Banach ideal. Then*

$$\left(\mathfrak{A}^{\Delta}\right)^{\text{inj}} \stackrel{1}{=} \mathfrak{P}_1 \circ \mathfrak{A}^{-1}$$

and

$$\left(\mathfrak{A}^{\Delta}\right)^{\text{sur}} \stackrel{1}{=} \mathfrak{A}^{-1} \circ \mathfrak{P}_1^{\text{d}} .$$

P r o o f. It is sufficient to prove the statement only for the injective hull. Since

$$\left(\mathfrak{A}^{\Delta}\right)^{\text{inj}} \circ \mathfrak{A} \stackrel{1}{\subseteq} \left(\mathfrak{A}^{\Delta} \circ \mathfrak{A}\right)^{\text{inj}} \stackrel{1}{\subseteq} \mathfrak{I}^{\text{inj}} \stackrel{1}{=} \mathfrak{P}_1 ,$$

it follows that $\left(\mathfrak{A}^{\Delta}\right)^{\text{inj}} \stackrel{1}{\subseteq} \mathfrak{P}_1 \circ \mathfrak{A}^{-1}$. To see the other inclusion, note that

$$\mathfrak{A}^{\Delta}(\,\cdot\,, Y_0) \stackrel{1}{=} \mathfrak{I} \circ \mathfrak{A}^{-1}(\,\cdot\,, Y_0)$$

holds for every Banach space Y_0 of which the dual has the metric approximation property (this follows by an direct application of [18], Lemma 10.2.6.). Hence,

$$\mathfrak{P}_1 \circ \mathfrak{A}^{-1} \stackrel{1}{=} \mathfrak{I}^{\text{inj}} \circ \mathfrak{A}^{-1} \stackrel{1}{\subseteq} \left(\mathfrak{I} \circ \mathfrak{A}^{-1}\right)^{\text{inj}} \stackrel{1}{=} \left(\mathfrak{A}^{\Delta}\right)^{\text{inj}} ,$$

and the proof is finished. □

A deeper investigation of relations between the Banach ideals $\left(\mathfrak{A}^{\Delta}, \mathbf{A}^{\Delta}\right)$ and $\left(\mathfrak{A}^*, \mathbf{A}^*\right)$ needs the help of an important local property, known as accessibility, which can be viewed as a local version of injectivity and surjectivity. All necesary details about accessibility and its applications can be found in [3], [5], [15], [16] and [17]. So let us recall:

(a) A quasi–Banach ideal $(\mathfrak{A}, \mathbf{A})$ is called *right–accessible*, if for all $(E, Y) \in \text{FIN} \times \text{BAN}$, operators $T \in \mathfrak{L}(E, Y)$ and $\varepsilon > 0$ there are $F \in \text{FIN}(Y)$ and $S \in \mathfrak{L}(E, F)$ so that $T = J_F^Y S$ and $\mathbf{A}(S) \leq (1 + \varepsilon)\mathbf{A}(T)$.

(b) $(\mathfrak{A}, \mathbf{A})$ is called *left–accessible*, if for all $(X, F) \in \text{BAN} \times \text{FIN}$, operators $T \in \mathfrak{L}(X, F)$ and $\varepsilon > 0$ there are $L \in \text{COFIN}(X)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = SQ_L^X$ and $\mathbf{A}(S) \leq (1 + \varepsilon)\mathbf{A}(T)$.

(c) A left–accessible and right–accessible quasi–Banach ideal is called *accessible*.

(d) $(\mathfrak{A}, \mathbf{A})$ is *totally accessible*, if for every finite rank operator $T \in \mathfrak{F}(X, Y)$ acting between Banach spaces X, Y and $\varepsilon > 0$ there are $(L, F) \in \text{COFIN}(X) \times \text{FIN}(Y)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = J_F^Y S Q_L^X$ and $\mathbf{A}(S) \leq (1 + \varepsilon)\mathbf{A}(T)$.

Let us recall the following important results on accessibility (for a detailed proof cf. [14], [17]):

Theorem 2.3. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a Banach ideal. Then* $\left(\mathfrak{A}^{*\Delta}, \mathbf{A}^{*\Delta}\right)$ *is always right–accessible. If in addition* $(\mathfrak{A}, \mathbf{A})$ *is maximal, then* $\left(\mathfrak{A}, \mathbf{A}\right)$ *is right–accessible if and only if* $\left(\mathfrak{A}^*, \mathbf{A}^*\right)$ *is left–accessible.*

Theorem 2.4. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal.*

(i) $(\mathfrak{A}, \mathbf{A})$ *is right–accessible if and only if* $\mathfrak{A}^* \circ \mathfrak{A} \overset{1}{\subseteq} \mathfrak{I}$.

(ii) $(\mathfrak{A}, \mathbf{A})$ *is totally accessible if and only if* $\mathfrak{A}^* \overset{1}{=} \mathfrak{A}^\Delta$.

Due to the existence of Banach spaces without the approximation property, we will see now that conjugate hulls are not "big enough" to contain such spaces. To this end, consider an arbitrary Banach ideal $(\mathfrak{A}, \mathbf{A})$, and let X be a Banach space so that $Id_X \in \mathfrak{A}^\Delta$ $\big(\text{i. e., } X \in \text{space}(\mathfrak{A}^\Delta)\big)$. Since $(\mathfrak{N}, \mathbf{N})$, the collection of all nuclear operators, is the smallest Banach ideal, it follows that $Id_X \in \mathfrak{N}^\Delta$ and $\mathbf{N}^\Delta(Id_X) \le \mathbf{A}^\Delta(Id_X)$. Hence, if $T \in \mathfrak{L}(X, X)$ is an arbitrary linear operator, it follows that $T = T Id_X \in \mathfrak{N}^\Delta(X, X)$ and $\mathbf{N}^\Delta(T) \le \|T\| \cdot \mathbf{N}^\Delta(Id_X) \le \|T\| \cdot \mathbf{A}^\Delta(Id_X)$. But this implies that

$$\mathfrak{L}(X, X) \;=\; \mathfrak{N}^\Delta(X, X)\,.$$

If $\mathfrak{A}$ contains the class $\mathfrak{I}$ of all integral operators (e. g., if $\mathfrak{A}$ is maximal or if $\mathfrak{A}$ is a conjugate of a quasi–Banach ideal), similar considerations lead to

$$\mathfrak{L}(X, X) \;=\; \mathfrak{I}^\Delta(X, X)\,,$$

and [11, Proposition 2.2.] now implies the following

Remark 2.5. Let $(\mathfrak{A}, \mathbf{A})$ be an arbitrary quasi–Banach ideal, and let X be a Banach space so that $X \in \text{space}\big(\mathfrak{A}^\Delta\big)$. If $\mathfrak{A}$ is normed, then X has the approximation property. If $\mathfrak{I} \subseteq \mathfrak{A}$, then X has the bounded approximation property.

Corollary 2.6. *Let* $(\mathfrak{A}, \mathbf{A})$ *be an arbitrary maximal Banach ideal so that there exists a Banach space in space*$(\mathfrak{A})$ *without the bounded approximation property, then* $\mathfrak{A}^\Delta$ $\big($*and therefore* $\mathfrak{A}^*\big)$ *cannot be totally accessible.*

P r o o f. Let X be a Banach space without the bounded approximation property so that $X \in \text{space}(\mathfrak{A})$. Assume, $\mathfrak{A}^\Delta$ is totally accessible, then

$$\mathfrak{A} \;\overset{1}{=}\; \mathfrak{A}^{**} \;\overset{1}{=}\; \mathfrak{A}^{\Delta *} \;\overset{1}{=}\; \mathfrak{A}^{\Delta\Delta}\,.$$

Since $\mathfrak{I} \overset{1}{=} \mathfrak{L}^\Delta \overset{1}{\subseteq} \mathfrak{A}^\Delta$, the previous remark leads to a contradiction[1]. □

3. Extension of finite rank operators and the principle of local reflexivity for operator ideals

Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. Then, $\mathfrak{A}^\Delta$ always is right–accessible (due to Theorem 2.3). The natural question whether $\mathfrak{A}^\Delta$ is left–accessible is still open[2] and leads to interesting and non–trivial results concerning the local structure of $\mathfrak{A}^\Delta$. Deeper investigations of the left–accessibility of $\mathfrak{A}^\Delta$ namely lead to a link with a principle of local reflexivity for operator ideals (a detailed discussion can be found

[1] Proposition 21.6 in [3] is a special case of this corollary.

[2] For minimal Banach ideals $(\mathfrak{A}, \mathbf{A})$, there exist counterexamples: The conjugate of $\mathfrak{A}_P^{\min}$ neither is right–accessible nor left–accessible (cf. [15]).

in [14] and [15]) which allows a transmission of the operator norm estimation in the classical principle of local reflexivity to the ideal norm $\mathbf{A}$. So let us recall the

Definition 3.1. Let E and Y be Banach spaces, E finite dimensional, $F \in \mathrm{FIN}(Y')$ and $T \in \mathfrak{L}(E, Y'')$. Let $(\mathfrak{A}, \mathbf{A})$ be a quasi–Banach ideal and $\epsilon > 0$. We say that the *principle of* $\mathfrak{A}$*–local reflexivity* (short: $\mathfrak{A}$*–LRP*) is satisfied, if there exists an operator $S \in \mathfrak{L}(E, Y)$ so that
(1) $\mathbf{A}(S) \leq (1+\epsilon) \cdot \mathbf{A}^{**}(T)$,
(2) $\langle Sx, y' \rangle = \langle y', Tx \rangle$ for all $(x, y') \in E \times F$,
(3) $j_Y Sx = Tx$ for all $x \in T^{-1}(j_Y(Y))$.

Although both, the quasi–Banach ideal $\mathfrak{A}$ and the 1–Banach ideal $\mathfrak{A}^{**}$ are involved, the asymmetry can be justified by the following statement which holds for arbitrary quasi–Banach ideals (see [15]):

Theorem 3.2. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a quasi–Banach ideal. Then the following statements are equivalent:*
(i) $\mathfrak{A}^{\Delta}$ *is left–accessible.*
(ii) $\mathfrak{A}^{**}(E, Y'') \cong \mathfrak{A}(E, Y)''$ *for all* $(E, Y) \in \mathrm{FIN} \times \mathrm{BAN}$.
(iii) *The* $\mathfrak{A}$*–LRP holds.*

Since every Banach ideal $(\mathfrak{A}, \mathbf{A})$ satisfies $\mathfrak{A}^{\Delta} \overset{1}{\subseteq} \mathfrak{A}^{*}$ and $\mathfrak{A}^{\Delta} \overset{\mathfrak{E}}{=} \mathfrak{A}^{*}$, we immediately obtain the following

Proposition 3.3. *If* $(\mathfrak{A}, \mathbf{A})$ *is a right–accessible maximal Banach ideal, then the* $\mathfrak{A}$*–LRP holds.*

So we do not know whether each maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ satisfies the $\mathfrak{A}$–LRP, like Pisier's counterexample $\big(\mathfrak{A}_P, \mathbf{A}_P\big)$ which neither is left–accessible nor right–accessible (cf. [3], 31.6). In particular, we would like to know whether the $\mathfrak{A}$–LRP even implies the right–accessibility of $(\mathfrak{A}, \mathbf{A})$. Due to Corollary 2.6, we already know that $\mathfrak{A}_P^{\Delta}$ cannot be totally accessible. Is it even true that $\big(\mathfrak{A}_P^{\Delta}\big)^{\mathrm{inj}} \overset{1}{=} \mathfrak{P}_1 \circ (\mathfrak{A}_P)^{-1}$ is not totally accessible? If this is the case, the $\mathfrak{A}_P$–LRP will be false.

One reason which leads to extreme persistent difficulties concerning the verification of the $\mathfrak{A}$–LRP for an arbitrary maximal Banach ideal $\mathfrak{A}$, is the behaviour of the bidual $\big(\mathfrak{A}^{\Delta}\big)^{\mathrm{dd}}$: although we know that in general $\big(\mathfrak{A}^{\Delta}\big)^{\mathrm{dd}}$ is accessible (see [14] and [15]) and that $\big(\mathfrak{A}^{\Delta}\big)^{\mathrm{dd}} \overset{1}{\subseteq} \mathfrak{A}^{\Delta}$, we do not know whether $\mathfrak{A}^{\Delta}(X, Y)$ and $\big(\mathfrak{A}^{\Delta}\big)^{\mathrm{dd}}(X, Y)$ coincide isometrically for all Banach spaces X and Y. If we allow in addition the approximation property of X or Y, then we may state the following

Lemma 3.4. *Let* $(\mathfrak{A}, \mathbf{A})$ *be an arbitrary maximal Banach ideal and* X, Y *be arbitrary Banach spaces. Then*

$$\mathfrak{A}^{\mathrm{d}\Delta}(X, Y) \overset{1}{=} \mathfrak{A}^{\Delta\mathrm{d}}(X, Y)$$

holds in each of the following two cases:

(i) X' *has the metric approximation property.*
(ii) Y' *has the metric approximation property and the* $\mathfrak{A}^{\mathrm{d}}$*-LRP is satisfied.*

Proof. Only the inclusion $\subseteq$ is not trivial. So, let $T \in \mathfrak{A}^{\mathrm{d}\Delta}(X,Y)$ be given. First, we consider the case (i). Due to proposition 2.3 of [11], it follows that in general

$$\mathfrak{A}^{\mathrm{d}\Delta} \overset{1}{\subseteq} \left(\mathfrak{A}^{\mathrm{d}}\right)^{-1} \circ \mathfrak{I} \overset{1}{=} \left(\mathfrak{A}^{\dashv}\right)^{\mathrm{d}},$$

so that $T' \in \mathfrak{A}^{\dashv}(Y',X')$, and $\mathbf{A}^{\dashv}(T') \leq \mathbf{A}^{\mathrm{d}\Delta}(T)$. Since X' has the metric approximation property we even obtain that $T' = Id_{X'}T' \in \mathfrak{I}^{\Delta} \circ \mathfrak{A}^{\dashv}(Y',X') \overset{1}{\subseteq} \mathfrak{A}^{\Delta}(Y',X')$, and case (i) is finished.

To prove case (ii), we have to proceed in a total different way. Let $L \in \mathfrak{F}(X',Y')$ be an arbitrary finite rank operator and $\epsilon > 0$. Since Y' has the metric approximation property, there exists a finite rank operator $A \in \mathfrak{F}(Y',Y')$ so that $L = AL$ and $\|A\| \leq 1+\epsilon$. Thanks to canonical factorization, we can find a finite dimensional space G and operators $A_1 \in \mathfrak{L}(Y',G'')$, $A_2 \in \mathfrak{L}(G'',Y')$ so that $A = A_2A_1$, $\|A_2\| \leq 1$ and $\|A_1\| \leq 1+\epsilon$. Now, look carefully at the composition of the two operators $A_1L \in \mathfrak{F}(X',G'')$ and $T'A_2 \in \mathfrak{L}(G'',X')$. Using exactly the same considerations as in [18, E.3.2.], the assumed $\mathfrak{A}^{\mathrm{d}}$–LRP implies the existence of an operator $\Lambda \in \mathfrak{L}(G',X)$ so that [3)]

$$\mathbf{A}^{\mathrm{d}}(\Lambda) \;\leq\; (1+\epsilon)\cdot\mathbf{A}^{\mathrm{d}}((A_1L)') \;=\; (1+\epsilon)\cdot\mathbf{A}(A_1L)$$

and $A_1LT'A_2 = \Lambda'T'A_2$. Since G is finite dimensional, we may represent A_2 as the dual of a finite rank operator $B_2 \in \mathfrak{F}(Y,G')$, and consequently it follows

$$\begin{aligned} |\mathrm{tr}(T'L)| \;=\; \left|\mathrm{tr}\big(A_1LT'A_2\big)\right| \;=\; \left|\mathrm{tr}\big(\Lambda'T'A_2\big)\right| \;&=\; \left|\mathrm{tr}\big(T\Lambda B_2\big)\right| \\ &\leq\; \mathbf{A}^{\mathrm{d}\Delta}(T)\cdot\mathbf{A}^{\mathrm{d}}(\Lambda) \\ &\leq\; (1+\epsilon)^2\cdot\mathbf{A}^{\mathrm{d}\Delta}(T)\cdot\mathbf{A}(L)\,. \end{aligned}$$

Hence, $T' \in \mathfrak{A}^{\Delta}(Y',X')$, and $\mathbf{A}^{\Delta}(T') \leq \mathbf{A}^{\mathrm{d}\Delta}(T)$, and case (ii) also is proved. □

A straightforward dualization of the previous lemma implies a result which we will use later again:

Corollary 3.5. *Let* $(\mathfrak{A},\mathbf{A})$ *be an arbitrary maximal Banach ideal and* X,Y *be arbitrary Banach spaces. Then*

$$\mathfrak{A}^{\Delta}(X,Y) \overset{1}{=} \left(\mathfrak{A}^{\Delta}\right)^{\mathrm{dd}}(X,Y)$$

holds in each of the following two cases:
(i) X'' *has the metric approximation property and the* $\mathfrak{A}^{\mathrm{d}}$*–LRP is satisfied.*
(ii) Y'' *has the metric approximation property and the* $\mathfrak{A}$*–LRP is satisfied.*

Now we consider the main tool in this paper, a factorization property for finite rank operators which had been introduced by Jarchow and Ott in their paper [11]. It not only turns out to be very useful for an investigation of local structures in operator

[3)] We only have to substitute the operator norm through the ideal norm $\mathbf{A}^{\mathrm{d}}$.

ideals; we will also use this factorization property to show that $\mathfrak{L}_\infty$ is not totally accessible — answering an open question of Defant and Floret (see [3], 21.12). So let us recall the definition of this factorization property and its implications:

Definition 3.6. (Jarchow/Ott.) Let $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ be arbitrary quasi–Banach ideals. Let $L \in \mathfrak{F}(X, Y)$ be an arbitrary finite rank operator between two Banach spaces X and Y. Given $\epsilon > 0$, we can find a Banach space Z and operators $A \in \mathfrak{A}(Z, Y)$, $B \in \mathfrak{B}(X, Z)$ so that $L = AB$ and

$$\mathbf{A}(A) \cdot \mathbf{B}(B) \ \leq \ (1+\epsilon) \cdot \mathbf{A} \circ \mathbf{B}(L)\,.$$

(i) If the operator A is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the *property* (I).
(ii) If the operator B is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the *property* (S).

Important examples are the following (see [11], Lemma 2.4.):
(a) If $\mathfrak{B}$ is injective, or if $\mathfrak{A}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).
(b) If $\mathfrak{A}$ is surjective, or if $\mathfrak{B}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).
Since $\mathfrak{L}_2 \circ \mathfrak{A}$ is injective for every quasi–Banach ideal $(\mathfrak{A}, \mathbf{A})$ (see [17], Lemma 5.1.), $\mathfrak{B} \circ \mathfrak{L}_2 \circ \mathfrak{A}$ therefore has the property (I) as well as the property (S), for all quasi–Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$. Such ideals are exactly those which contain $\mathfrak{L}_2$ as factor — in the sense of [11].
The next statement will be also useful for our further investigatons (see [11], 2.5.):

Proposition 3.7. *Let* $(\mathfrak{A}, \mathbf{A})$ *and* $(\mathfrak{B}, \mathbf{B})$ *be arbitrary quasi–Banach ideals. Then*
(i) $(\mathfrak{A} \circ \mathfrak{B})^\Delta \stackrel{1}{=} \mathfrak{B}^{-1} \circ \mathfrak{A}^\Delta$, *if* $\mathfrak{A} \circ \mathfrak{B}$ *has the property* (I).
(ii) $(\mathfrak{A} \circ \mathfrak{B})^\Delta \stackrel{1}{=} \mathfrak{B}^\Delta \circ \mathfrak{A}^{-1}$, *if* $\mathfrak{A} \circ \mathfrak{B}$ *has the property* (S).
In both cases (i) *and* (ii)*, the inclusion* $\stackrel{1}{\subseteq}$ *holds in general — without any assumption on the ideals* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Next, we will see how the property (I) of the product ideal $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ influences the structure of operator ideals of type $\mathfrak{A}^{\text{inj}*} \stackrel{1}{=} \diagdown \mathfrak{A}^*$ and their conjugates. To this end, first note that for all Banach spaces X, Y and $X \stackrel{1}{\hookrightarrow} Z$, every operator $T \in \big(\mathfrak{A}^{\text{inj}}\big)^*(X, Y) \stackrel{1}{=} \diagdown \mathfrak{A}^*(X, Y)$ satisfies the following extension property: Given $\epsilon > 0$, there exists an operator $\widetilde{T} \in \diagdown \mathfrak{A}^*(Z, Y'')$ so that $j_Y T = \widetilde{T} J_X^Z$ and $\diagdown \mathbf{A}^*\big(\widetilde{T}\big) \leq (1+\epsilon) \cdot \diagdown \mathbf{A}^*(T)$ (see [9], Satz 7.14). In particular, such an extension holds for all finite rank operators. However, we then cannot be sure that $\widetilde{T}$ is also as a finite rank operator. Here, property (I) comes into play — in the following sense:

Theorem 3.8. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a Banach ideal so that* $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ *has the property* (I)*. Let* $\epsilon > 0$*,* X *and* Y *be arbitrary Banach spaces and* $L \in \mathfrak{F}(Y, X)$*. Let* Z *be a Banach space which contains* Y *isometrically. Then there exists a finite rank operator* $U \in \mathfrak{F}(Z, X'')$ *so that* $j_X L = U J_Y^Z$ *and*

$$\big(\mathbf{A}^{\text{inj}}\big)^*(U) \ \leq \ (1+\epsilon) \cdot \big(\mathbf{A}^{\text{inj}}\big)^*(L)\,.$$

If in addition, the $\mathfrak{A}^*$–LRP is satisfied, then U even can be chosen to be a finite rank operator with range in X and $L = UJ_Y^Z$.

Proof. WLOG, we may assume that $(\mathfrak{A}, \mathbf{A})$ is maximal. Let $L \in \mathfrak{F}(Y, X)$ be an arbitrary finite rank operator between arbitrarily given Banach spaces X and Y, and set $(\mathfrak{B}, \mathbf{B}) := \big(\mathfrak{A}^{\text{inj}}, \mathbf{A}^{\text{inj}}\big)$. Let $\epsilon > 0$. Since $\mathfrak{B}^* \overset{1}{=} (\mathfrak{A}^* \circ \mathfrak{L}_\infty)^{reg}$ (cf. [17]), there exists a Banach space W and operators $A \in \mathfrak{A}^*(W, X'')$, $B \in \mathfrak{L}_\infty(Y, W)$ so that $j_X L = AB$ and

$$\mathbf{A}^*(A) \cdot \mathbf{L}_\infty(B) \;\leq\; (1+\epsilon) \cdot \mathbf{B}^*(L)\,.$$

Due to the assumed property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, we even may assume that A is a finite rank operator. Further, we also may choose a Borel–Radon measure μ and operators $R \in \mathfrak{L}(L_\infty(\mu), W'') \overset{1}{=} \mathfrak{L}_\infty(L_\infty(\mu), W'')$, $S \in \mathfrak{L}(Y, L_\infty(\mu))$ so that $j_W B = RS$ and

$$\mathbf{L}_\infty(R) \cdot \|S\| \;\leq\; (1+\epsilon) \cdot \mathbf{L}_\infty(B)$$

(cf. [3], 20.12). Due to the metric extension property of $L_\infty(\mu)$, the operator S can be extended to an operator $\widetilde{S} \in \mathfrak{L}(Z, L_\infty(\mu))$ so that $S = \widetilde{S} J_Y^Z$ and $\big\|\widetilde{S}\big\| = \|S\|$. If we also take into account that $Id_{X''} = j'_{X'} j_{X''}$, then we obtain the following factorization of $j_X L$:

$$j_X L \;=\; j'_{X'}(j_{X''} j_X L) \;=\; j'_{X'}\big(A'' R \widetilde{S} J_Y^Z\big)\,.$$

Therefore, $U := j'_{X'} A'' R \widetilde{S} \in \mathfrak{F}(Z, X'')$ is the desired finite rank operator, and the factorization further shows that

$$\mathbf{A}^* \circ \mathbf{L}_\infty(U) \;\leq\; (1+\epsilon)^2 \cdot \mathbf{B}^*(L)\,,$$

and the first part of our theorem is proven.

Now let us assume that in addition the $\mathfrak{A}^*$–LRP is satisfied. Since Y embeds isometrically into $Y^\infty = l_\infty(B_{Y'})$, the previous considerations (in particular) imply the existence of a finite rank operator $V \in \mathfrak{F}(Y^\infty, X'')$, so that $j_X L = V J_Y$ and $\mathbf{B}^*(V) \leq (1+\epsilon) \cdot \mathbf{B}^*(L)$. Due to the metric approximation property of the dual of Y^∞, we can find a finite dimensional subspace F in Y^∞ and an operator $B \in \mathfrak{L}(Y^\infty, F)$ so that $\|B\| \leq 1+\epsilon$ and $V = WB$ where $W := V J_F^{Y^\infty} \in \mathfrak{L}(F, X'')$. Due to the assumed $\mathfrak{A}^*$–LRP (which obviously implies the $\mathbf{B}^*$–LRP), we even can find an operator $W_0 \in \mathfrak{L}(F, X)$ so that

$$\mathbf{B}^*(W_0) \;\leq\; (1+\epsilon) \cdot \mathbf{B}^*(W) \;\leq\; (1+\epsilon)^2 \cdot \mathbf{B}^*(L)$$

and

$$Wx \;=\; j_X W_0 x \quad \text{for all} \quad x \in W^{-1}(j_X(X))\,.$$

Since for every $y \in Y$, $x = BJ_Y y \in F$ and $Wx = WBJ_Y y = VJ_Y y = j_X L y \in j_X(X)$, it therefore follows that

$$j_X L y \;=\; j_X W_0 B J_Y y \quad \text{for all} \quad y \in Y\,.$$

Hence, $L = W_0 B J_Y$ and $\big(\mathbf{A}^{\mathrm{inj}}\big)^*(W_0 B) \le (1+\epsilon)^3 \cdot \big(\mathbf{A}^{\mathrm{inj}}\big)^*(L)$. Since Y^∞ has the metric extension property, we can factorize J_Y as $J_Y = \widetilde{J} J_Y^Z$ so that $\widetilde{J} \in \mathfrak{L}(Z, Y^\infty)$, $\|\widetilde{J}\| = 1$, and $U_0 := W_0 B \widetilde{J} \in \mathfrak{L}(Z, X)$ is our desired finite rank operator. □

Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal and $\big(\mathfrak{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}}\big)$ its injective hull. Thinking carefully about the previous statement, one might guess a strong relationship between the conjugate of $\big(\mathfrak{A}^{\mathrm{inj}}\big)^*$ and the injective hull of $\mathfrak{A}^{*\Delta}$ — involving the $\mathfrak{A}^*$–LRP and further accessibility conditions. Indeed, this is the case:

Theorem 3.9. *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal so that the $\mathfrak{A}^*$–LRP is satisfied. Then*

$$\mathfrak{A}^{*\Delta\mathrm{inj}} \overset{1}{\subseteq} \big(\mathfrak{A}^{*\Delta\mathrm{inj}}\big)^{\mathrm{dd}}. \tag{3.1}$$

If in addition, $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I)*, then*

$$\big(\mathfrak{A}^{*\Delta\mathrm{inj}}\big)^{\mathrm{dd}} \overset{1}{=} \mathfrak{A}^{*\Delta\mathrm{inj}} \overset{1}{=} \mathfrak{P}_1 \circ \big(\mathfrak{A}^*\big)^{-1} \overset{1}{=} \mathfrak{A}^{\mathrm{inj}*\Delta} \overset{1}{=} \big(\mathfrak{A}^{\mathrm{inj}*\Delta}\big)^{\mathrm{dd}}. \tag{3.2}$$

Proof. First, let the $\mathfrak{A}^*$–LRP be satisfied. Let $T \in \mathfrak{A}^{*\Delta\mathrm{inj}}(X,Y)$ be given and X, Y be arbitrary Banach spaces. Due to Corollary 3.5 and the assumed validity of the $\mathfrak{A}^*$–LRP, it follows that $J_Y'' T'' = (J_Y T)'' \in \mathfrak{A}^{*\Delta}(X'', (Y^\infty)'')$ and

$$\mathbf{A}^{*\Delta}(J_Y'' T'') \le \mathbf{A}^{*\Delta}(J_Y T) = \big(\mathbf{A}^{*\Delta}\big)^{\mathrm{inj}}(T).$$

Since $J_Y'' : Y'' \overset{1}{\hookrightarrow} (Y^\infty)''$ is an isometric embedding (cf. [18], B.3.9.), the metric extension property of $(Y'')^\infty$ implies the existence of an operator $\widetilde{J} \in \mathfrak{L}((Y^\infty)'', (Y'')^\infty)$ so that $J_{Y''} = \widetilde{J} J_Y''$ and $\|\widetilde{J}\| = 1$. Hence, $T'' \in \big(\mathfrak{A}^{*\Delta}\big)^{\mathrm{inj}}(X'', Y'')$ and

$$\big(\mathbf{A}^{*\Delta}\big)^{\mathrm{inj}}(T'') \le \big(\mathbf{A}^{*\Delta}\big)^{\mathrm{inj}}(T),$$

which implies the inclusion (3.1). To prove (3.2), note, that the second isometric identity already has been proven in this paper (see Proposition 2.2). Recalling that always

$$\mathfrak{A}^{\mathrm{inj}*\Delta} \overset{1}{\subseteq} \big(\mathfrak{A}^{*\Delta}\big)^{\mathrm{inj}},$$

we only have to prove the inclusion

$$\big(\mathfrak{A}^{*\Delta\mathrm{inj}}\big)^{\mathrm{dd}} \overset{1}{\subseteq} \mathfrak{A}^{\mathrm{inj}*\Delta}$$

— given the property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$. To this end, let $T \in \big(\mathfrak{A}^{*\Delta\mathrm{inj}}\big)^{\mathrm{dd}}(X,Y)$ be given, with arbitrarily chosen Banach spaces X and Y, and put $(\mathfrak{B}, \mathbf{B}) := \big(\mathfrak{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}}\big)$. Since $\mathfrak{B}^{*\Delta}$ is regular (see Proposition 2.1), we only have to show that $j_Y T \in \mathfrak{B}^{*\Delta}(X, Y'')$ and

$$\mathbf{B}^{*\Delta}(j_Y T) \le \big(\mathbf{A}^{*\Delta}\big)^{\mathrm{inj}}(T'').$$

So, let $L \in \mathfrak{F}(Y'', X)$ be an arbitrary finite rank operator — considered as an element of $\mathfrak{B}^*(Y'', X)$. Due to the assumed property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, Theorem 3.8 shows us the existence of a finite rank operator $V \in \mathfrak{F}((Y'')^\infty, X'')$ so that $j_X L = V J_{Y''}$ and

$$\mathbf{A}^*(V) \;\leq\; (1+\epsilon) \cdot \mathbf{B}^*(L)\,.$$

Hence,

$$\begin{aligned} \bigl|\mathrm{tr}\bigl(j_Y T L\bigr)\bigr| \;=\; \bigl|\mathrm{tr}\bigl(T'' j_X L\bigr)\bigr| \;=\; \bigl|\mathrm{tr}\bigl(T'' V J_{Y''}\bigr)\bigr| \;&=\; \bigl|\mathrm{tr}\bigl(J_{Y''} T'' V\bigr)\bigr| \\ &\leq\; \mathbf{A}^{*\Delta}(J_{Y''} T'') \cdot \mathbf{A}^*(V) \\ &\leq\; (1+\epsilon) \cdot \bigl(\mathbf{A}^{*\Delta}\bigr)^{\mathrm{inj}}(T'') \cdot \mathbf{B}^*(L)\,, \end{aligned}$$

which implies that $j_Y T \in \mathfrak{B}^{*\Delta}(X, Y'')$ and $\mathbf{B}^{*\Delta}(j_Y T) \leq (\mathbf{A}^{*\Delta})^{\mathrm{inj}}(T'')$. Summing up all the previous steps in our proof, we have shown that

$$\mathfrak{A}^{*\Delta\mathrm{inj}} \;\overset{1}{=}\; \mathfrak{A}^{\mathrm{inj}*\Delta} \;\overset{1}{=}\; \bigl(\mathfrak{A}^{*\Delta\mathrm{inj}}\bigr)^{\mathrm{dd}}$$

which obviously implies (3.2), and the proof is finished. □

Corollary 3.10. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal so that* $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ *has the property* (I) *and* $\mathfrak{A}^{\mathrm{inj}}$ *is totally accessible. Then even* $\bigl(\mathfrak{A}^{\mathrm{inj}}\bigr)^*$ *is totally accessible.*

Proof. Let $\mathfrak{A}^{\mathrm{inj}}$ be totally accessible. Then $\mathfrak{A}^{\mathrm{inj}}$ in particular is left–accessible which implies that

$$\mathfrak{A}^{\mathrm{inj}} \circ \mathfrak{A}^* \circ \mathfrak{L}_\infty \;\overset{1}{\subseteq}\; \mathfrak{A}^{\mathrm{inj}} \circ (\mathfrak{A}^{\mathrm{inj}})^* \;\overset{1}{\subseteq}\; \mathfrak{I}\,.$$

Hence $\mathfrak{A}^{\mathrm{inj}} \circ \mathfrak{A}^* \overset{1}{\subseteq} \mathfrak{I} \circ \mathfrak{L}_\infty^{-1} \overset{1}{=} \mathfrak{P}_1$. Due to the assumed property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, Proposition 2.2 and the proof of the previous statement therefore imply that

$$\mathfrak{A}^{\mathrm{inj}} \;\overset{1}{=}\; \bigl(\mathfrak{A}^{\mathrm{inj}}\bigr)^{\mathrm{dd}} \;\overset{1}{\subseteq}\; \bigl(\mathfrak{P}_1 \circ (\mathfrak{A}^*)^{-1}\bigr)^{\mathrm{dd}} \;\overset{1}{=}\; \bigl((\mathfrak{A}^{*\Delta})^{\mathrm{inj}}\bigr)^{\mathrm{dd}} \;\overset{1}{\subseteq}\; \bigl(\mathfrak{A}^{\mathrm{inj}}\bigr)^{*\Delta}.$$

Since $(\mathfrak{A}, \mathbf{A})$ is maximal, then even

$$\mathfrak{A}^{\mathrm{inj}} \;\overset{1}{=}\; \bigl(\mathfrak{A}^{\mathrm{inj}}\bigr)^{*\Delta},$$

and Theorem 2.4 finishes the proof. □

Theorem 3.9 and Corollary 3.10 imply interesting consequences. First, we observe that any injective maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ which contains Banach spaces without the bounded approximation property (such as $(\mathfrak{C}_2, \mathbf{C_2})$) cannot be totally accessible if in addition $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I) (due to Corollary 2.6 and Corollary 3.10).

Concerning Banach spaces of cotype 2, we only have to add some of our own techniques to a deep result of Pisier, to prove the next statement[4)]:

[4)] Note that it is not necessary to assume that X resp. Y has the Gordon–Lewis property (cf. [4], Theorem 17.12).

Theorem 3.11. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal so that* $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ *has the property* (I) *and* $\mathfrak{A}^{\mathrm{inj}}$ *is totally accessible. Let* X *and* Y *be Banach spaces so that both* X' *and* Y *have cotype* 2. *Then*

$$\mathfrak{A}^{\mathrm{inj}}(X,Y) \subseteq \mathfrak{L}_2(X,Y)\,,$$

and

$$\mathbf{L}_2(T) \;\le\; (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^{\mathrm{inj}}(T)$$

for all operators $T \in \mathfrak{A}^{\mathrm{inj}}(X,Y)$.

Proof. Let X and Y be as above and put $C := (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}}$. Then, [20, Theorem 4.9] tells us, that any finite rank operator $L \in \mathfrak{F}(Y,X)$ satisfies

$$\mathbf{N}(L) \;\le\; C \cdot \mathbf{D}_2(L)\,.$$

Hence,

$$\mathfrak{N}^\Delta(X,Y) \;\subseteq\; \mathfrak{D}_2^\Delta(X,Y) \;\stackrel{1}{=}\; \mathfrak{L}_2(X,Y)\,,$$

and

$$\mathbf{L}_2(T) \;\le\; C \cdot \mathbf{N}^\Delta(T)$$

for all operators $T \in \mathfrak{N}^\Delta(X,Y)$. Since $\mathfrak{N} \stackrel{1}{\subseteq} \big(\mathfrak{A}^{\mathrm{inj}}\big)^*$, we therefore obtain

$$\big(\mathfrak{A}^{\mathrm{inj}}\big)^{*\Delta}(X,Y) \;\subseteq\; \mathfrak{L}_2(X,Y)\,,$$

and

$$\mathbf{L}_2(T) \;\le\; C \cdot \big(\mathbf{A}^{\mathrm{inj}}\big)^{*\Delta}(T)$$

for all operators $T \in \big(\mathfrak{A}^{\mathrm{inj}}\big)^{*\Delta}(X,Y)$. Given our assumptions on $\mathfrak{A}$, Corollary 3.10 reveals that $\mathfrak{A}^{\mathrm{inj}} \stackrel{1}{\subseteq} \big(\mathfrak{A}^{\mathrm{inj}}\big)^{*\Delta}$, and the claim follows. □

In relation to the $\mathfrak{A}$–LRP, the property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ leads to the following

Proposition 3.12. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a Banach ideal so that* $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ *has the property* (I). *If space*$(\mathfrak{A})$ *contains a Banach space* X_0 *so that* X_0 *has the bounded approximation property but* X_0'' *has not, then the* $\mathfrak{A}^*$*–LRP cannot be satisfied.*

Proof. Assume, that the statement is false and hence the $\mathfrak{A}^*$–LRP is satisfied. Since X_0 has the bounded approximation property, $Id_{X_0} \in \mathfrak{I}^\Delta(X_0,X_0)$ and $c := \mathbf{I}^\Delta\big(Id_{X_0}\big) < \infty$. By definition of $\mathfrak{I}^\Delta$ and of the adjoint $\mathfrak{A}^*$, one immediately derives the inclusion

$$\mathfrak{A} \circ \mathfrak{I}^\Delta \circ \mathfrak{A}^* \stackrel{1}{\subseteq} \mathfrak{I}\,,$$

so that in particular

$$\mathfrak{A}(X_0,X_0) \;\subseteq\; \mathfrak{I} \circ \big(\mathfrak{A}^*\big)^{-1}(X_0,X_0) \;\stackrel{1}{\subseteq}\; \mathfrak{P}_1 \circ \big(\mathfrak{A}^*\big)^{-1}(X_0,X_0) \;\stackrel{1}{=}\; \mathfrak{A}^{*\Delta\mathrm{inj}}(X_0,X_0)$$

and

$$\mathbf{A}^{*\Delta\mathrm{inj}}\big(Id_{X_0}\big) \;\le\; c \cdot \mathbf{A}\big(Id_{X_0}\big)\,.$$

Hence, due to the assumed property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, Theorem 3.9 implies that even $X_0'' \in \text{space}\big(\mathfrak{A}^{\text{inj}*\Delta}\big)$ and

$$\mathbf{A}^{\text{inj}*\Delta}\big(Id_{X_0''}\big) \stackrel{1}{=} \mathbf{A}^{\text{inj}*\Delta}\big(Id_{X_0}''\big) \leq c \cdot \mathbf{A}\big(Id_{X_0}\big).$$

But this would imply that $X_0'' \in \text{space}\big(\mathfrak{I}^\Delta\big)$, leading to the conclusion that X_0'' would have the bounded approximation property — with constant $c \cdot \mathbf{A}\big(Id_{X_0}\big)$, which is a contradiction. □

Our next application considers the question of Defant and Floret (see [3], 21.12) whether $\mathfrak{L}_\infty$ is totally accessible or not. We are able to show that $\mathfrak{L}_\infty$ is not totally accessible, and the idea of the proof is the following: Assuming the opposite, leads to the property (I) for a suitable class of quasi–Banach ideals of type $\mathfrak{A}^* \circ \mathfrak{L}_\infty$. On the other hand, there exists a well–known left–accessible candidate $\mathfrak{A}$ so that $\big(\mathfrak{A}^{\text{inj}}\big)^*$ is not totally accessible, which is a contradiction to Corollary 3.10. To prepare the steps carefully, we first state a fact which is of its own interest:

Lemma 3.13. *Let* $(\mathfrak{A}, \mathbf{A})$ *and* $(\mathfrak{B}, \mathbf{B})$ *be arbitrary quasi–Banach ideals so that*
(i) $\mathfrak{A} \circ \mathfrak{B}$ *has the property* (S),
(ii) $\mathfrak{B}$ *is totally accessible.*
Then $\mathfrak{A} \circ \mathfrak{B}$ *is left–accessible and has the property* (I).

Proof. Let X, Y be arbitrary Banach spaces and $L \in \mathfrak{F}(X, Y)$ be an arbitrary finite rank operator. Given $\epsilon > 0$, there exists a Banach space Z and operators $A \in \mathfrak{A}(Z, Y)$, $B \in \mathfrak{B}(X, Z)$ so that $L = AB$ and

$$\mathbf{A}(A) \cdot \mathbf{B}(B) \leq (1 + \epsilon) \cdot \mathbf{A} \circ \mathbf{B}(L).$$

Due to the property (S) of $\mathfrak{A} \circ \mathfrak{B}$, we may assume that B is of finite rank. Hence, since $\mathfrak{B}$ is totally accessible, there exist $K \in \text{COFIN}(X)$, $E \in \text{FIN}(Z)$ and an operator $\Gamma \in \mathfrak{L}(X/K, E)$ so that $B = J_E^Z \Gamma Q_K^X$ and

$$\mathbf{B}(\Gamma) \leq (1 + \epsilon) \cdot \mathbf{B}(B).$$

Therefore, $L = A_0 \Gamma Q_K^X$ where $A_0 := A J_E^Z \in \mathfrak{F}(E, Y)$ and

$$\mathbf{A} \circ \mathbf{B}(A_0 \Gamma) \leq \mathbf{A}(A_0) \cdot \mathbf{B}(\Gamma) \leq (1 + \epsilon)^2 \cdot \mathbf{A} \circ \mathbf{B}(L),$$

and the claim follows. □

Obviously, similar arguments allow a transfer of property (S) to property (I), and we obtain the "(I)–version":

Lemma 3.14. *Let* $(\mathfrak{A}, \mathbf{A})$ *and* $(\mathfrak{B}, \mathbf{B})$ *be arbitrary quasi–Banach ideals so that*
(i) $\mathfrak{A} \circ \mathfrak{B}$ *has the property* (I),
(ii) $\mathfrak{A}$ *is totally accessible.*
Then $\mathfrak{A} \circ \mathfrak{B}$ *has the property* (S) *and is right–accessible.*

Now, we are well prepared to investigate the total accessibility of $\mathfrak{L}_\infty$:

Theorem 3.15. *The maximal Banach ideals $\mathfrak{L}_\infty \sim g_\infty$ and $\mathfrak{L}_1 \sim w_1$ are not totally accessible.*

Proof. Since $\mathfrak{L}_1 \stackrel{1}{=} \mathfrak{L}_\infty^{\mathrm{d}}$, we only have to prove the claim for $\mathfrak{L}_\infty$. Assume the opposite. Consider the maximal Banach ideal $\mathfrak{A} := \mathfrak{L}_1 \stackrel{1}{=} \left(\mathfrak{P}_1^{\mathrm{d}}\right)^*$. Since

$$\left(\mathfrak{A}^*\right)^{\mathrm{sur}} \stackrel{1}{=} \left(\mathfrak{P}_1^{\mathrm{d}}\right)^{\mathrm{sur}} \stackrel{1}{=} \left(\mathfrak{P}_1^{\mathrm{inj}}\right)^{\mathrm{d}} \stackrel{1}{=} \mathfrak{P}_1^{\mathrm{d}} \stackrel{1}{=} \mathfrak{A}^*,$$

it follows that $\mathfrak{A}^*$ is surjective, so that $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (S). Due to Lemma 3.13, the assumed total accessibility of $\mathfrak{L}_\infty$ even leads to the property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, and Corollary 3.10 implies that $\left(\mathfrak{A}^{\mathrm{inj}}\right)^* \stackrel{1}{=} \left(\mathfrak{L}_1^{\mathrm{inj}}\right)^*$ is totally accessible. On the other hand, [3, Corollary 21.6.2] tells us that the adjoint of $\mathfrak{L}_1^{\mathrm{inj}}$ cannot be totally accessible (because of the existence of subspaces of l_1 without the approximation property), and we obtain a contradiction. □

Since $\mathfrak{L}_\infty \sim g_\infty$ and $\mathfrak{L}_1 \sim w_1$ are not totally accessible, their respective adjoints $\mathfrak{P}_1$ and $\mathfrak{P}_1^{\mathrm{d}}$ cannot have the m.a.p. factorization property — answering a question of [5].

Corollary 3.16. $\mathfrak{L}_\infty \circ \mathfrak{L}_\infty$ *neither has the property* (I) *nor the property* (S) *and is not regular. In particular,* $\mathfrak{L}_\infty \stackrel{1}{\neq} \mathfrak{L}_\infty \circ \mathfrak{L}_\infty$.

Proof. First, assume that $\mathfrak{L}_\infty \circ \mathfrak{L}_\infty$ has the property (S). Then, Proposition 3.7 implies that

$$\mathfrak{L}_\infty^\Delta \circ \mathfrak{L}_\infty^{-1} \stackrel{1}{=} \left(\mathfrak{L}_\infty \circ \mathfrak{L}_\infty\right)^\Delta.$$

Since $\mathfrak{P}_1$ is left–accessible, it follows that

$$\mathfrak{P}_1 \circ \mathfrak{L}_\infty \stackrel{1}{=} \mathfrak{P}_1 \circ \mathfrak{P}_1^* \stackrel{1}{\subseteq} \mathfrak{I} \stackrel{1}{=} \mathfrak{L}^\Delta \stackrel{1}{\subseteq} \mathfrak{L}_\infty^\Delta,$$

and hence $\mathfrak{P}_1 \stackrel{1}{\subseteq} \left(\mathfrak{L}_\infty \circ \mathfrak{L}_\infty\right)^\Delta$. But this inclusion further implies that

$$\mathfrak{P}_1^{\mathrm{d}} \stackrel{1}{\subseteq} \left(\mathfrak{L}_\infty \circ \mathfrak{L}_\infty\right)^{\mathrm{d}\Delta} \stackrel{1}{=} \left(\left(\mathfrak{L}_\infty \circ \mathfrak{L}_\infty\right)^{reg}\right)^{\mathrm{d}\Delta} \stackrel{1}{=} \mathfrak{L}_\infty^{\mathrm{d}\Delta} \stackrel{1}{=} \mathfrak{L}_1^\Delta,$$

$\left(\text{since } \mathfrak{L}_\infty \stackrel{1}{=} \left(\mathfrak{L}_\infty \circ \mathfrak{L}_\infty\right)^{reg}\right)$ and we obtain the contradiction $\mathfrak{L}_1^* \stackrel{1}{=} \mathfrak{L}_1^\Delta$ (since $\mathfrak{L}_1$ is not totally accessible). Since $\mathfrak{L}_\infty \stackrel{1}{=} \mathfrak{P}_1^*$ is not totally accessible, Corollary 3.10 implies that $\mathfrak{P}_1^* \circ \mathfrak{L}_\infty \stackrel{1}{=} \mathfrak{L}_\infty \circ \mathfrak{L}_\infty$ cannot have the property (I). Obviously, $\mathfrak{L} \circ \mathfrak{L}_\infty \stackrel{1}{=} \mathfrak{L}_\infty$ has the property (S), and the proof is finished. □

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