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On normed products of operator ideals which contain $\mathfrak{L}_2$ as a factor

By

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Abstract. We investigate quasi-Banach operator ideal products $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ which contain $(\mathfrak{L}_2, \mathbf{L}_2)$ as a factor. In particular, we ask for conditions which guarantee that $\mathbf{A} \circ \mathbf{B}$ is even a norm if each factor of the product is a 1-Banach ideal. In doing so, we reveal the strong influence of the existence of such a norm in relation to the accessibility of the product ideal and the structure of its factors.

1. Introduction. This paper is devoted to an investigation of the normability of operator ideal products which contain $(\mathfrak{L}_2, \mathbf{L}_2)$ as a factor (where $\mathfrak{L}_2$ denotes the class of all operators which factor through a Hilbert space). It seems that 1-Banach operator ideal products play a fundamental role in the search for maximal Banach ideals which do not satisfy a transfer of the norm estimation in the classical principle of local reflexivity to their ideal norm (cf. [14]). This problem (which still is open) originated from the objective to facilitate the search for a non-accessible maximal normed Banach ideal (which is the same as a non-accessible finitely generated tensor norm in the sense of Grothendieck) (cf. [10]). Later, in 1993, Pisier constructed a counterexample (cf. [2, 31.6.]). Since each right-accessible maximal Banach ideal $(\mathfrak{A}, \mathbf{A})$ even satisfies such a *principle of local reflexivity for operator ideals*, Pisier's counterexample of a non-accessible maximal Banach ideal naturally lead to the search for counterexamples of maximal Banach ideals $(\mathfrak{A}_0, \mathbf{A}_0)$ for which the conjugate $(\mathfrak{A}_0^\Delta, \mathbf{A}_0^\Delta)$ is not left-accessible, implying surprising relations between the existence of a norm on product operator ideals of type $\mathfrak{B} \circ \mathfrak{L}_2$, the extension of finite rank operators with respect to a suitable operator ideal norm and the principle of local reflexivity for operator ideals (cf. [14]). The basic objects, connecting these different aspects, are product operator ideals with the property (I) and the property (S), introduced by Jarchow and Ott (see [8]). In the widest sense, a product operator ideal $\mathfrak{A} \circ \mathfrak{B}$ has the property (I), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = (\mathfrak{A} \cap \mathfrak{F}) \circ \mathfrak{B}$$

and the property (S), if

$$(\mathfrak{A} \circ \mathfrak{B}) \cap \mathfrak{F} = \mathfrak{A} \circ (\mathfrak{B} \cap \mathfrak{F}),$$

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(where $\mathfrak{F}$ denotes the class of all finite rank operators) so that each finite rank operator in $\mathfrak{A} \circ \mathfrak{B}$ is the composition of two operators, one of which is of finite rank. Since each operator ideal which contains $\mathfrak{L}_2$ as a factor, has both, the property (I) and the property (S), Hilbert space factorization crystallized out as a fundamental key in these investigations.

2. The framework. In this section, we introduce the basic notation and terminology which we will use throughout in this paper. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard. We refer the reader to the monographs [2], [3] and [15] for the necessary background in operator ideal theory and the related terminology. Infinite dimensional Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are denoted throughout by W, X, Y and Z in contrast to the letters E, F and G which are used for finite dimensional Banach spaces only. Denote for given Banach spaces X and Y

$$\begin{aligned} \mathrm{FIN}(X) &:= \{E \subseteqq X : E \in \mathrm{FIN}\} \text{ and} \\ \mathrm{COFIN}(X) &:= \{L \subseteqq X : X/L \in \mathrm{FIN}\}, \end{aligned}$$

where FIN stands for the class of all finite dimensional Banach spaces. The space of all operators (continuous linear maps) from X to Y is denoted by $\mathfrak{L}$(X,Y), and for the identity operator on X, we write Id_X. The collection of all finite rank (resp. approximable) operators from X to Y is denoted by $\mathfrak{F}$(X,Y) (resp. $\overline{\mathfrak{F}}$(X,Y)), and $\mathfrak{E}$(X,Y) indicates the collection of all operators, acting between finite dimensional Banach spaces X and Y (elementary operators). The dual of a Banach space X is denoted by X', and X'' denotes its bidual $(X')'$. If $T \in \mathfrak{L}$(X,Y) is an operator, we indicate that it is a metric injection by writing $T : X \overset{1}{\hookrightarrow} Y$, and if it is a metric surjection, we write $T : X \overset{1}{\twoheadrightarrow} Y$. If X is a Banach space, E a finite dimensional subspace of X and K a finite codimensional subspace of X, then $B_X := \{x \in X : \|x\| \leqq 1\}$ denotes the closed unit ball, $J_E^X : E \overset{1}{\hookrightarrow} X$ the canonical metric injection and $Q_K^X : X \overset{1}{\twoheadrightarrow} X/K$ the canonical metric surjection. Finally, $T' \in \mathfrak{L}(Y', X')$ denotes the dual operator of $T \in \mathfrak{L}$(X,Y).

If $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ are given quasi-Banach ideals, we will use throughout the shorter notation $(\mathfrak{A}^{\mathrm{d}}, \mathbf{A}^{\mathrm{d}})$ for the dual ideal and the abbreviation $\mathfrak{A} \overset{1}{=} \mathfrak{B}$ for the isometric equality $(\mathfrak{A}, \mathbf{A}) = (\mathfrak{B}, \mathbf{B})$. We write $\mathfrak{A} \subseteqq \mathfrak{B}$ if, regardless of the Banach spaces X and Y, we have $\mathfrak{A}(X, Y) \subseteqq \mathfrak{B}(X, Y)$. The metric inclusion $(\mathfrak{A}, \mathbf{A}) \subseteqq (\mathfrak{B}, \mathbf{B})$ is often shortened by $\mathfrak{A} \overset{1}{\subseteqq} \mathfrak{B}$.

Given quasi-Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$, let $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ be the corresponding product ideal and $(\mathfrak{A} \circ \mathfrak{B}^{-1}, \mathbf{A} \circ \mathbf{B}^{-1})$ (resp. $(\mathfrak{A}^{-1} \circ \mathfrak{B}, \mathbf{A}^{-1} \circ \mathbf{B})$) the corresponding "right-quotient" (resp. "left-quotient"). Important examples are $(\mathfrak{A}^{\min}, \mathbf{A}^{\min}) := (\overline{\mathfrak{F}} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}, \| \cdot \| \circ \mathbf{A} \circ \| \cdot \|)$ (the *minimal kernel* of $(\mathfrak{A}, \mathbf{A})$) and $(\mathfrak{A}^{\max}, \mathbf{A}^{\max}) := (\overline{\mathfrak{F}}^{-1} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}^{-1}, \| \cdot \|^{-1} \circ \mathbf{A} \circ \| \cdot \|^{-1})$ (the *maximal hull* of $(\mathfrak{A}, \mathbf{A})$). $(\mathfrak{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}})$ denotes the *injective hull* of $\mathfrak{A}$, the unique smallest injective quasi-Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$, and $(\mathfrak{A}^{\mathrm{sur}}, \mathbf{A}^{\mathrm{sur}})$, the *surjective hull* of $\mathfrak{A}$, is the unique smallest surjective quasi-Banach ideal which contains $(\mathfrak{A}, \mathbf{A})$.

In addition to the maximal Banach ideal $(\mathfrak{L}, \|\cdot\|)$ we mainly will be concerned with the maximal Banach ideals $(\mathfrak{I}, \mathbf{I})$ (integral operators), $(\mathfrak{L}_2, \mathbf{L}_2)$ (Hilbertian operators), $(\mathfrak{D}_2, \mathbf{D}_2) \overset{1}{=} (\mathfrak{L}_2^*, \mathbf{L}_2^*) \overset{1}{=} \mathfrak{P}_2^{\mathrm{d}} \circ \mathfrak{P}_2$ (2-dominated operators), $(\mathfrak{P}_p, \mathbf{P}_p)$ (absolutely p-summing operators), $1 \leqq p \leqq \infty$, $(\mathfrak{L}_\infty, \mathbf{L}_\infty) \overset{1}{=} (\mathfrak{P}_1^*, \mathbf{P}_1^*)$ and $(\mathfrak{L}_1, \mathbf{L}_1) \overset{1}{=} (\mathfrak{P}_1^{*\mathrm{d}}, \mathbf{P}_1^{*\mathrm{d}})$.

Since we will use them throughout in this paper, let us recall the important notions of the conjugate operator ideal (cf. [5], [8] and [11]) and the adjoint operator ideal (all details can be found in the standard references [2] and [15]). Let $(\mathfrak{A}, \mathbf{A})$ be a quasi-Banach ideal.

- Let $\mathfrak{A}^\Delta(X, Y)$ be the set of all $T \in \mathfrak{L}(X, Y)$ which satisfy
$$\mathbf{A}^\Delta(T) := \sup\{|\mathrm{tr}(TL)| : L \in \mathfrak{F}(Y, X), \mathbf{A}(L) \leqq 1\} < \infty.$$
Then a Banach ideal $(\mathfrak{A}^\Delta, \mathbf{A}^\Delta)$ is obtained (here, $\mathrm{tr}(\cdot)$ denotes the usual trace for finite rank operators). It is called the *conjugate ideal* of $(\mathfrak{A}, \mathbf{A})$.
- Let $\mathfrak{A}^*(X, Y)$ be the set of all $T \in \mathfrak{L}(X, Y)$ which satisfy
$$\mathbf{A}^*(T) := \sup\{|\mathrm{tr}(TJ_E^X S Q_K^Y\}| : E \in \mathrm{FIN}(X), K \in \mathrm{COFIN}(Y), \mathbf{A}(S) \leqq 1\} < \infty.$$
Then a Banach ideal $(\mathfrak{A}^*, \mathbf{A}^*)$ is obtained. It is called the *adjoint operator ideal* of $(\mathfrak{A}, \mathbf{A})$.

A deeper investigation of relations between the Banach ideals $(\mathfrak{A}^\Delta, \mathbf{A}^\Delta)$ and $(\mathfrak{A}^*, \mathbf{A}^*)$ needs the help of an important local property, known as accessibility, which can be viewed as a local version of injectivity and surjectivity. All necesary details about accessibility of operator ideals and its applications can be found in [2], [11], [12] and [13]. So let us recall:

- A quasi-Banach ideal $(\mathfrak{A}, \mathbf{A})$ is called *right-accessible*, if for all $(E, Y) \in \mathrm{FIN} \times \mathrm{BAN}$, operators $T \in \mathfrak{L}(E, Y)$ and $\varepsilon > 0$ there are $F \in \mathrm{FIN}(Y)$ and $S \in \mathfrak{L}(E, F)$ so that $T = J_F^Y S$ and $\mathbf{A}(S) \leqq (1+\varepsilon)\mathbf{A}(T)$.
- $(\mathfrak{A}, \mathbf{A})$ is called *left-accessible*, if for all $(X, F) \in \mathrm{BAN} \times \mathrm{FIN}$, operators $T \in \mathfrak{L}(X, F)$ and $\varepsilon > 0$ there are $L \in \mathrm{COFIN}(X)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = SQ_L^X$ and $\mathbf{A}(S) \leqq (1+\varepsilon)\mathbf{A}(T)$.
- A left-accessible and right-accessible quasi-Banach ideal is called *accessible*.
- $(\mathfrak{A}, \mathbf{A})$ is *totally accessible*, if for every finite rank operator $T \in \mathfrak{F}(X, Y)$ acting between Banach spaces X, Y and $\varepsilon > 0$ there are $(L, F) \in \mathrm{COFIN}(X) \times \mathrm{FIN}(Y)$ and $S \in \mathfrak{L}(X/L, F)$ so that $T = J_F^Y S Q_L^X$ and $\mathbf{A}(S) \leqq (1+\varepsilon)\mathbf{A}(T)$.

Let us recall the following important results on accessibility (for a detailed proof cf. [2], [10] and [13]):

Theorem 2.1. *Let $(\mathfrak{A}, \mathbf{A})$ be a Banach ideal. Then $(\mathfrak{A}^{\min}, \mathbf{A}^{\min})$ always is accessible and $(\mathfrak{A}^{*\Delta}, \mathbf{A}^{*\Delta})$ is right-accessible. If in addition $(\mathfrak{A}, \mathbf{A})$ is maximal, then $(\mathfrak{A}, \mathbf{A})$ is right-accessible if and only if $(\mathfrak{A}^*, \mathbf{A}^*)$ is left-accessible.*

Theorem 2.2. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal.*

(i) $(\mathfrak{A}, \mathbf{A})$ *is right-accessible if and only if* $\mathfrak{A}^* \circ \mathfrak{A} \overset{1}{\subseteqq} \mathfrak{I}$.
(ii) $(\mathfrak{A}, \mathbf{A})$ *is totally accessible if and only if* $\mathfrak{A}^* \overset{1}{=} \mathfrak{A}^{\Delta}$.

Pisier's counterexample $(\mathfrak{A}_P, \mathbf{A}_P)$ shows the existence of maximal Banach ideals which neither are left nor right-accessible (cf. [2], 31.6). However, accessibility of a quasi-Banach ideal at least can be transmitted to its regular hull:

Proposition 2.1. *Let* $(\mathfrak{A}, \mathbf{A})$ *be an arbitrary quasi-Banach ideal. If* $(\mathfrak{A}, \mathbf{A})$ *is right-accessible* (*resp. totally-accessible*), *then the regular hull* $(\mathfrak{A}^{\mathrm{reg}}, \mathbf{A}^{\mathrm{reg}})$ *is also right-accessible* (*resp. totally-accessible*).

Proof. Let $\epsilon > 0$, X, Y be Banach spaces and $T \in F(X, Y)$ an arbitrary finite rank operator. Assume that $\mathfrak{A}$ is totally accessible or that $X \in FIN$ and $\mathfrak{A}$ is right-accessible. In both cases, there exists a finite dimensional Banach space $F \in FIN(Y'')$ and an operator $S \in \mathfrak{L}(X, F)$, so that $j_Y T = J_F^{Y''} S$ and

$$\mathbf{A}(S) < (1+\epsilon) \cdot \mathbf{A}(j_Y T) = (1+\epsilon) \cdot \mathbf{A}^{\mathrm{reg}}(T).$$

Due to the classical principle of local reflexivity for linear operators, there exists an operator $W \in \mathfrak{L}(F, Y)$ so that $\|W\| < 1+\epsilon$ and $j_Y W z = J_F^{Y''} z$ for all $z \in F$ which satisfy $J_F^{Y''} z \in j_Y(Y)$. Let $x \in X$ and put $z := Sx$. Then $J_F^{Y''} z = j_Y T x \in j_Y(Y)$, which therefore implies that $j_Y W S x = J_F^{Y''} z = j_Y T x$. Now, factor W canonically through a finite dimensional subspace G of Y so that $W = J_G^Y U$ and $\|U\| < 1+\epsilon$. Consequently, $T = WS = J_G^Y(US)$, and

$$\mathbf{A}^{\mathrm{reg}}(US) < (1+\epsilon)^2 \cdot \mathbf{A}^{\mathrm{reg}}(T).$$

Hence, $\mathfrak{A}^{\mathrm{reg}}$ is right-accessible (in each of the both cases). In the case of $\mathfrak{A}$ being totally accessible, the operator S even can be chosen as $S = S_0 Q_K^X$, where $K \in COFIN(X)$ and $S_0 \in \mathfrak{L}(X/K, F)$ so that

$$\mathbf{A}(S_0) < (1+\epsilon) \cdot \mathbf{A}^{\mathrm{reg}}(T),$$

and the proof is finished. $\square$

3. Normed operator ideal products. Let $(\mathfrak{A}, \mathbf{A})$ be a p-normed Banach ideal ($0 < p \leqq 1$) and $(\mathfrak{B}, \mathbf{B})$ be a q-normed Banach ideal ($0 < q \leqq 1$). Then the product $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ is a r-normed Banach ideal, where $1/r := 1/p + 1/q$ (see [15], 7.1.2). Even if $p = 1$ and $q = 1$, $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ in general is a $1/2$-Banach ideal only; $\mathbf{A} \circ \mathbf{B}$ need not to be a norm. However, if one of the operator ideals is closed (such as e.g., $(\overline{\mathfrak{F}}, \|\cdot\|)$, $(\mathfrak{K}, \|\cdot\|)$ or $(\mathfrak{W}, \|\cdot\|)$), then we may formulate a positive result (cf. [1]):

Proposition 3.1. *Let* $(\mathfrak{A}, \mathbf{A})$ *and* $(\mathfrak{B}, \mathbf{B})$ *be two quasi-Banach ideals. Let* $0 < p \leqq 1$. *Then, in each of the following cases,* $(\mathfrak{A} \circ \mathfrak{B}, \mathbf{A} \circ \mathbf{B})$ *is a p-Banach ideal:*

(i) $(\mathfrak{A}, \mathbf{A})$ *is a p-Banach ideal and* $(\mathfrak{B}, \mathbf{B}) = (\mathfrak{B}, \|\cdot\|)$ *is closed,*
(ii) $(\mathfrak{A}, \mathbf{A}) = (\mathfrak{A}, \|\cdot\|)$ *is closed and* $(\mathfrak{B}, \mathbf{B})$ *is a p-Banach ideal.*

Proof. It is sufficient to prove the case (i); (ii) follows similarily. So let $(\mathfrak{A}, \mathbf{A})$ be p-normed and $(\mathfrak{B}, \|\cdot\|)$ be closed. Let X and Y be arbitrary Banach spaces and $T_1, T_2 \in \mathfrak{A} \circ \mathfrak{B}(X, Y)$. It remains to show that

$$(\mathbf{A} \circ \|\cdot\|)^p(T_1 + T_2) \leqq (\mathbf{A} \circ \|\cdot\|)^p(T_1) + (\mathbf{A} \circ \|\cdot\|)^p(T_2).$$

Let $\epsilon > 0$. Then there exist Banach spaces Z_1, Z_2 and operators $R_1 \in \mathfrak{A}(Z_1, Y)$, $R_2 \in \mathfrak{A}(Z_2, Y)$, $S_1 \in \mathfrak{B}(X, Z_1)$, $S_2 \in \mathfrak{B}(X, Z_2)$ so that $T_1 = R_1 S_1$, $T_2 = R_2 S_2$, $\|S_1\| \leqq 1$, $\|S_2\| \leqq 1$, $\mathbf{A}(R_1) \leqq (1+\epsilon)\cdot(\mathbf{A} \circ \|\cdot\|)(T_1)$ and $\mathbf{A}(R_2) \leqq (1+\epsilon)\cdot(\mathbf{A} \circ \|\cdot\|)(T_2)$. We now consider the Banach space $W := l_\infty(Z_1, Z_2)$ consisting of all elements $(z_1, z_2) \in Z_1 \times Z_2$ so that $\|(z_1, z_2)\|_\infty := \max(\|z_1\|, \|z_2\|) < \infty$. Let $J_i : Z_i \overset{1}{\hookrightarrow} W$ be the canonical injections and $Q_i : W \overset{1}{\twoheadrightarrow} Z_i$ the corresponding canonical surjections $(i = 1, 2)$. Then $S := J_1 S_1 + J_2 S_2 \in \mathfrak{B}(X, W)$ and $\|Sx\|_\infty = \|(S_1 x, S_2 x)\|_\infty = \max(\|S_1 x\|, \|S_2 x\|) \leqq \|x\|$ for all $x \in X$. Hence, $\|S\| \leqq 1$. Put $R := R_1 Q_1 + R_2 Q_2$. Then $R \in \mathfrak{A}(W, Y)$ and $(\mathbf{A}(R))^p \leqq (\mathbf{A}(R_1))^p + (\mathbf{A}(R_2))^p$. The construction therefore implies that $T_1 + T_2 = RS$ and $(\mathbf{A} \circ \|\cdot\|)^p(T_1 + T_2) = (\mathbf{A} \circ \|\cdot\|)^p(RS) \leqq (\mathbf{A}(R)\cdot\|S\|)^p \leqq (\mathbf{A}(R))^p \leqq (\mathbf{A}(R_1))^p + (\mathbf{A}(R_2))^p \leqq (1+\epsilon)^p \cdot ((\mathbf{A} \circ \|\cdot\|)^p(T_1) + (\mathbf{A} \circ \|\cdot\|)^p(T_2))$, and the proof is finished. □

An immediate (non-trivial) consequence is the

Corollary 3.1. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a p-Banach ideal* $(0 < p \leqq 1)$. *Then* $(\mathfrak{A}^{\min}, \mathbf{A}^{\min})$ *is also a p-Banach ideal.*

Unfortunately, we still cannot present explicit sufficient criteria which show the existence of (an equivalent) ideal norm on product ideals in the general case. It seems to be much more easier to show that a certain product ideal cannot be a normed one by using arguments which involve trace ideals and the ideal of nuclear operators (the smallest Banach ideal). Even more holds: if $\mathfrak{A} \circ \mathfrak{L}_2$ is a 1-Banach ideal for certain operator ideals $\mathfrak{A}$, then $\mathfrak{A} \circ \mathfrak{L}_2$ *is not* right-accessible (cf. Theorem 3.4)! To investigate more carefully the general case, we recall an important factorization property for finite rank operators which had been introduced by Jarchow and Ott in their paper [8]. It not only turns out to be very useful for an investigation of local structures in (product) operator ideals; this factorization property was used as the main tool to show that $\mathfrak{L}_\infty$ and $\mathfrak{L}_1$ are *not* totally accessible – answering an open question of Defant and Floret (see [2], 21.12 and [14]). So, let us recall the definition of this factorization property and its implications:

Definition 3.1. (Jarchow/Ott). Let $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$ be arbitrary quasi-Banach ideals. Let $L \in \mathfrak{F}(X, Y)$ an arbitrary finite rank operator between two Banach spaces

X and Y. Given $\epsilon > 0$, we can find a Banach space Z and operators $A \in \mathfrak{A}(Z, Y)$, $B \in \mathfrak{B}(X, Z)$ so that $L = AB$ and

$$\mathbf{A}(A) \cdot \mathbf{B}(B) \leqq (1+\epsilon) \cdot (\mathbf{A} \circ \mathbf{B})(L).$$

(i) If the operator A is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).
(ii) If the operator B is of finite rank, we say that $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Important examples are the following (see [8], Lemma 2.4.):

- If $\mathfrak{B}$ is injective, or if $\mathfrak{A}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (I).
- If $\mathfrak{A}$ is surjective, or if $\mathfrak{B}$ contains $\mathfrak{L}_2$ as a factor, then $\mathfrak{A} \circ \mathfrak{B}$ has the property (S).

Since $\mathfrak{L}_2 \circ \mathfrak{A}$ is injective for every quasi-Banach ideal $(\mathfrak{A}, \mathbf{A})$ (see [13], Lemma 5.1), $\mathfrak{B} \circ \mathfrak{L}_2 \circ \mathfrak{A}$ therefore has the property (I) as well as the property (S), for all quasi-Banach ideals $(\mathfrak{A}, \mathbf{A})$ and $(\mathfrak{B}, \mathbf{B})$. Such ideals are exactly those which contain $\mathfrak{L}_2$ as factor – in the sense of [8].

Theorem 3.1. *Let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal. Then both, the maximal $\frac{1}{2}$-Banach ideal $\mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2$ and the maximal $\frac{1}{2}$-Banach ideal $(\mathfrak{A} \circ \mathfrak{L}_2)^{\text{inj}}$ are totally accessible.*

Proof. Since every Hilbert space has the metric approximation property and since $\mathfrak{A}^{\text{inj}} \overset{1}{=} (\mathfrak{A}^{\text{inj}})^{**}$ is right-accessible, an easy calculation shows that

$$\mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2 \overset{1}{=} (\mathfrak{A}^{\text{inj}})^{*\Delta} \circ \mathfrak{L}_2. \tag{1}$$

Since $(\mathfrak{A}^{\text{inj}})^{*\Delta}$ is right-accessible, the total accessibility of $\mathfrak{L}_2$ and the property (S) of the product ideal $(\mathfrak{A}^{\text{inj}})^{*\Delta} \circ \mathfrak{L}_2$ even imply that $(\mathfrak{A}^{\text{inj}})^{*\Delta} \circ \mathfrak{L}_2$ is totally accessible (cf. [14, Proposition 4.1]). Hence, $\mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2$ is totally accessible (due to (1)), and in particular we obtain that $(\mathfrak{A} \circ \mathfrak{L}_2)^{\text{inj}} \overset{1}{=} (\mathfrak{A}^{\text{inj}} \circ \mathfrak{L}_2)^{\text{inj}}$ is totally accessible. $\square$

Now, let $(\mathfrak{A}, \mathbf{A})$ be a maximal Banach ideal so that $\mathbf{L}_2 \circ \mathbf{A}$ even is a norm on the (maximal) product ideal $(\mathfrak{L}_2 \circ \mathfrak{A}, \mathbf{L}_2 \circ \mathbf{A})$. Then $\mathfrak{A}^* \overset{1}{\subseteqq} (\mathfrak{L}_2 \circ \mathfrak{A})^* \overset{1}{\subseteqq} \mathfrak{L}_\infty$ (cf. [13, Proposition 5.1]) and $\mathfrak{L}_\infty \overset{1}{=} \mathfrak{P}_1^\Delta \overset{1}{\subseteqq} \mathfrak{N}^\Delta$. Given Banach spaces X and Y so that both, X' and Y have cotype 2, [16, Theorem 4.9] tells us, that any finite rank operator $L \in \mathfrak{F}(Y, X)$ satisfies

$$\mathbf{N}(L) \leqq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{D}_2(L).$$

Hence,

$$\mathfrak{N}^\Delta(X, Y) \subseteqq \mathfrak{D}_2^\Delta(X, Y) \overset{1}{=} \mathfrak{L}_2(X, Y),$$

and we have proven a rather surprising fact (revealing the strong influence of a *norm* on an operator ideal product):

Theorem 3.2. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal so that the product ideal* $(\mathfrak{L}_2 \circ \mathfrak{A}, \mathbf{L}_2 \circ \mathbf{A})$ *is normed. Let* X *and* Y *be arbitrary Banach spaces so that both,* X' *and* Y *have cotype* 2*. Then*

$$\mathfrak{A}^*(X, Y) \overset{1}{\subseteqq} (\mathfrak{L}_2 \circ \mathfrak{A})^*(X, Y) \subseteqq \mathfrak{L}_2(X, Y)$$

and

$$\begin{aligned} \mathbf{L}_2(T) &\leqq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot (\mathbf{L}_2 \circ \mathbf{A})^*(T) \\ &\leqq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^*(T) \end{aligned}$$

for all operators $T \in \mathfrak{A}^*(X, Y)$.

To maintain the previous statement, even a permutation of the factors $\mathfrak{A}$ and $\mathfrak{L}_2$ in the product $\mathfrak{L}_2 \circ \mathfrak{A}$ is allowed:

Theorem 3.3. *Let* $(\mathfrak{A}, \mathbf{A})$ *be a maximal Banach ideal so that the product ideal* $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ *is normed. Let* X *and* Y *be arbitrary Banach spaces so that both,* X' *and* Y *have cotype* 2*. Then*

$$\mathfrak{A}^*(X, Y) \overset{1}{\subseteqq} (\mathfrak{A} \circ \mathfrak{L}_2)^*(X, Y) \subseteqq \mathfrak{L}_2(X, Y)$$

and

$$\begin{aligned} \mathbf{L}_2(T) &\leqq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot (\mathbf{A} \circ \mathbf{L}_2)^*(T) \\ &\leqq (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}} \cdot \mathbf{A}^*(T) \end{aligned}$$

for all operators $T \in \mathfrak{A}^*(X, Y)$.

Proof. Let $(\mathfrak{A}, \mathbf{A})$ and X, Y be as before and let $\mathfrak{A} \circ \mathfrak{L}_2$ be normed. Then $\mathfrak{A} \overset{1}{=} \mathfrak{A}^{\mathrm{dd}}$, and $\mathfrak{A} \circ \mathfrak{L}_2$ is a maximal (and therefore a regular) Banach ideal (cf. [14, Lemma 4.3]). Since the injective $\frac{1}{2}$-Banach ideal $\mathfrak{L}_2 \circ \mathfrak{A}^{\mathrm{d}}$ is also regular (cf. [13, Lemma 5.1]), an easy calculation shows that

$$(\mathfrak{A} \circ \mathfrak{L}_2)^{\mathrm{d}} \overset{1}{=} \mathfrak{L}_2 \circ \mathfrak{A}^{\mathrm{d}}$$

and[1)]

$$\mathfrak{A} \circ \mathfrak{L}_2 \overset{1}{=} (\mathfrak{L}_2 \circ \mathfrak{A}^{\mathrm{d}})^{\mathrm{d}}.$$

Since $\mathbf{A} \circ \mathbf{L}_2$ is a norm, $(\mathbf{A} \circ \mathbf{L}_2)^{\mathrm{d}}$ obviously is a norm too. Hence, if we apply the previous theorem to the normed product ideal $(\mathfrak{A} \circ \mathfrak{L}_2)^{\mathrm{d}} \overset{1}{=} \mathfrak{L}_2 \circ \mathfrak{A}^{\mathrm{d}}$, we obtain

$$\mathfrak{A}^{*\mathrm{d}}(X, Y) \overset{1}{\subseteqq} (\mathfrak{L}_2 \circ \mathfrak{A}^{\mathrm{d}})^*(X, Y) \overset{1}{=} (\mathfrak{A} \circ \mathfrak{L}_2)^{*\mathrm{d}}(X, Y) \subseteqq \mathfrak{L}_2(X, Y),$$

[1)] In particular, it follows that $\mathfrak{A} \circ \mathfrak{L}_2$ is surjective (cf. [15, 8.5.9.]).

and

$$\mathbf{L}_2(T) \leqq C \cdot (\mathbf{A} \circ \mathbf{L}_2)^*(T') \leqq C \cdot \mathbf{A}^*(T')$$

for all operators $T \in \mathfrak{A}^{*\mathrm{d}}(X, Y)$ (where $C := (2\mathbf{C}_2(X') \cdot \mathbf{C}_2(Y))^{\frac{3}{2}}$). Now, since Y has the same coptype as its bidual $(Y')'$ with identical cotype constants (cf. [3, Corollary 11.9]), the proof is finished. $\square$

Let $(\mathfrak{A}, \mathbf{A})$ be a given ultrastable quasi-Banach ideal so that $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ is right-accessible. Our aim is to show that in this case $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ and $(\mathfrak{L}_2 \circ \mathfrak{A}^*, \mathbf{L}_2 \circ \mathbf{A}^*)$ *both together* cannot be normed. To this end, we need a lemma which is of its own interest:

Lemma 3.1. *Let* $(\mathfrak{A}_0, \mathbf{A}_0)$ *be a maximal Banach ideal so that space*$(\mathfrak{A}_0)$ *contains a Banach space without the approximation property. Then there does not exist a maximal Banach ideal* $(\mathfrak{C}, \mathbf{C})$ *so that* $\mathfrak{C} \circ \mathfrak{L}_\infty$ *has the property* (I) *and* $\mathfrak{C} \subseteqq \mathfrak{A}_0^{-1} \circ \mathfrak{P}_1$.

P r o o f. Assume that the statement is false. Then there exists a (maximal) Banach ideal $(\mathfrak{A}, \mathbf{A})$ so that $\mathfrak{A}_0 \subseteqq \mathfrak{P}_1 \circ (\mathfrak{A}^*)^{-1} \stackrel{1}{=} (\mathfrak{A}^{*\Delta})^{\mathrm{inj}}$. Due to the assumed property (I) of $\mathfrak{A}^* \circ \mathfrak{L}_\infty$, the proof of Theorem 3.4 in [14] shows that even $((\mathfrak{A}^{*\Delta})^{\mathrm{inj}})^{\mathrm{dd}} \stackrel{1}{\subseteqq} (\mathfrak{A}^{\mathrm{inj}})^{*\Delta} \stackrel{1}{\subseteqq} \mathfrak{N}^\Delta$. Since $\mathfrak{A}_0$ was assumed to be a maximal Banach ideal, we therefore obtain $\mathfrak{A}_0 \stackrel{1}{=} \mathfrak{A}_0^{\mathrm{dd}} \stackrel{1}{\subseteqq} \mathfrak{N}^\Delta$ which is a contradiction. $\square$

Corollary 3.2. *Let* $(\mathfrak{A}_0, \mathbf{A}_0)$ *be a maximal Banach ideal so that space*$(\mathfrak{A}_0)$ *contains a Banach space without the approximation property. If* $(\mathfrak{A}_0^{-1} \circ \mathfrak{P}_1) \circ \mathfrak{L}_\infty$ *has the property* (I), $\mathfrak{A}_0$ *is not left-accessible.*

Theorem 3.4. *Let* $(\mathfrak{B}, \mathbf{B})$ *be an ultrastable quasi-Banach ideal so that* $\mathfrak{B} \subseteqq \mathfrak{L}_\infty$. *If* $\mathfrak{B} \circ \mathfrak{L}_2$ *is right-accessible,* $\mathfrak{B} \circ \mathfrak{L}_2$ *cannot be a* 1*-Banach ideal.*

P r o o f. Assume that the statement is false and put $\mathfrak{B}_0 := (\mathfrak{L}_\infty \circ \mathfrak{L}_2)^*$ and $\mathfrak{A} := (\mathfrak{B} \circ \mathfrak{L}_2)^*$. Then

$$\mathfrak{A}^* \stackrel{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{\max} \stackrel{1}{=} (\mathfrak{B} \circ \mathfrak{L}_2)^{\mathrm{reg}} \stackrel{1}{=} \mathfrak{B}^{\mathrm{reg}} \circ \mathfrak{L}_2$$

is right-accessible (cf. [14, Proposition 2.3]) and contains $\mathfrak{L}_2$ as a factor so that in particular $\mathfrak{A}^* \circ \mathfrak{L}_\infty$ has the property (I). Since $\mathfrak{B} \subseteqq \mathfrak{L}_\infty$,

$$\mathfrak{B}_0 \circ \mathfrak{A}^* \subseteqq \mathfrak{A} \circ \mathfrak{A}^* \stackrel{1}{\subseteqq} \mathfrak{I} \stackrel{1}{\subseteqq} \mathfrak{P}_1,$$

and it follows that $\mathfrak{A}^* \subseteqq \mathfrak{B}_0^{-1} \circ \mathfrak{P}_1$. Since $Id_P \in \mathfrak{B}_0$ (cf. [14, Proposition 4.4]), Lemma 3.1 leads to a contradiction. $\square$

Now let us assume that $(\mathfrak{B}, \mathbf{B})$ is even is a *maximal Banach ideal* so that $\mathfrak{B} \subsetneqq \mathfrak{L}_\infty$. If $\mathfrak{B} \circ \mathfrak{L}_2$ were normed, then $\mathfrak{B} \circ \mathfrak{L}_2$ would be a maximal *and* surjective Banach ideal, implying that $\mathfrak{P}_1^{\mathrm{d}} \overset{1}{=} \mathfrak{I}^{\mathrm{sur}} \overset{1}{=} (\mathfrak{N}^{\max})^{\mathrm{sur}} \overset{1}{\subseteqq} \mathfrak{B} \circ \mathfrak{L}_2$. Hence,

$$(\mathfrak{B} \circ \mathfrak{L}_2)^* \overset{1}{\subseteqq} \mathfrak{P}_1^{\mathrm{d}*} \overset{1}{=} \mathfrak{L}_1.$$

Since $\mathfrak{B} \subsetneqq \mathfrak{L}_\infty$, it even follows that $\mathfrak{P}_1 \overset{1}{=} \mathfrak{L}_\infty^* \subsetneqq (\mathfrak{B} \circ \mathfrak{L}_2)^* \overset{1}{\subseteqq} \mathfrak{L}_1$ which is a contradiction (cf. [2, 27.2.]). So, in this case we obtain a stronger result:

Theorem 3.5. *Let $(\mathfrak{B}, \mathbf{B})$ be a maximal Banach ideal so that $\mathfrak{B} \subsetneqq \mathfrak{L}_\infty$. Then $\mathfrak{B} \circ \mathfrak{L}_2$ cannot be a 1-Banach ideal.*

Corollary 3.3. *Let $(\mathfrak{A}, \mathbf{A})$ be a quasi-Banach ideal. If $(\mathfrak{A}, \mathbf{A})$ is a maximal Banach ideal or if $(\mathfrak{A}, \mathbf{A})$ is ultrastable and $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ is right-accessible, then $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ and $(\mathfrak{L}_2 \circ \mathfrak{A}^*, \mathbf{L}_2 \circ \mathbf{A}^*)$ both together cannot be normed.*

Proof. Let $(\mathfrak{A} \circ \mathfrak{L}_2, \mathbf{A} \circ \mathbf{L}_2)$ be a 1-Banach ideal. Then $\mathfrak{A} \not\subseteqq \mathfrak{L}_\infty$. If the injective quasi-Banach ideal $(\mathfrak{L}_2 \circ \mathfrak{A}^*, \mathbf{L}_2 \circ \mathbf{A}^*)$ were also a normed one, then we would obtain $\mathfrak{P}_1 \overset{1}{\subseteqq} \mathfrak{L}_2 \circ \mathfrak{A}^* \overset{1}{\subseteqq} \mathfrak{A}^*$ (cf. [13], Proposition 5.1) – a contradiction. □

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