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Upper bounds for Grothendieck constants, quantum correlation matrices and CCP functions. (English) Zbl 07820894

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The Grothendieck inequality states the following: There is an absolute constant K such that for all real or complex matrices (of any size) satisfying

$$\Big|\sum_{i,j} a_{ij} s_i t_j\Big| \le 1 \quad \text{whenever } |s_i|, |t_j| \le 1$$

one has

$$\Big|\sum_{i,j} a_{ij} \langle u_i, v_j\rangle\Big| \le K$$

whenever u_i, v_j are vectors in a Hilbert space with $\|u_i\|, \|v_j\| \le 1$. A superb survey on the Grothendieck inequality is [*G. Pisier*, Bull. Am. Math. Soc., New Ser. 49, No. 2, 237–323 (2012; Zbl 1244.46006)].

The above is a reformulation, due to *J. Lindenstrauss* and *A. Pełczyński* [Stud. Math. 29, 275–326 (1968; Zbl 0183.40501)], of Grothendieck's "Théorème fondamental de la théorie métrique des produits tensoriels" from [*A. Grothendieck*, Bol. Soc. Mat. São Paulo 8, 1–79 (1956; Zbl 0074.32303)], which states the equivalence of certain tensor norms; it is the only result in this tersely written 79-page paper that he found worthy of a proof. At the end of his paper, Grothendieck states six problems that have meanwhile been solved with the exception of Problem 3 which asks for the best constant in the above inequality. He correctly predicts, "Il n'est d'ailleurs pas certain que ces constantes soient les mêmes dans la 'théorie réelle' et la 'théorie complexe' ", and the best constant in the real (resp. complex) case will be denoted by $K_G^{\mathbb{R}}$ (resp. $K_G^{\mathbb{C}}$).

It is this problem to which the present monograph contributes. The author presents a strategy how to obtain upper bounds for K_G and implements it to reproduce Krivine's bound in the real case, i.e.,

$$K_G^{\mathbb{R}} \le \frac{\pi}{2\log(1+\sqrt{2})} = \frac{\operatorname{arcsin}(1)}{\operatorname{arsinh}(1)} = 1.782\dots$$

[*J. L. Krivine*, Adv. Math. 31, 16–30 (1979; Zbl 0413.46054)] and at least partially the result from [*M. Braverman* et al., Forum Math. Pi 1, Paper No. e4, 42 p. (2013; Zbl 1320.15016)] (see also [*J.-L. Krivine*, "A note about Grothendieck's constant", Preprint, arXiv:2306.00995]) that the above inequality is strict, without offering an explicit smaller bound. In the complex case, he almost reproduces Haagerup's bound

$$\Big(\frac{4}{\pi} \le\Big)\ K_G^{\mathbb{C}} \le 1.404\dots\ \Big(< \frac{\pi}{2} \le K_G^{\mathbb{R}}\Big)$$

from [*U. Haagerup*, Isr. J. Math. 60, No. 2, 199–224 (1987; Zbl 0646.46019)]. Incidentally, the best known lower bounds for K_G available in print still seem to be the ones above, due to Grothendieck himself [loc. cit.] – modulo a correction of an oversight concerning $K_G^{\mathbb{C}}$; unpublished works due to Davie and Krivine claim sharper bounds.

Very roughly, the author's approach is this. First, the Grothendieck inequality is rewritten in terms of correlation matrices (Prop. 3.2, page 49). Then, certain entrywise transformations of matrices are introduced: A function f is called completely correlation preserving (CCP for short) if for each correlation matrix (s_{ij}), the matrix $(f(s_{ij}))$ is a correlation matrix as well. (In this work, a correlation matrix is a positive semidefinite matrix having 1's on the diagonal; they are characterised in various ways, cf. Lemma 3.1 on page 40.) Schoenberg has shown that real continuous CCP functions are exactly those having a representation $f(x) = \sum_{n=0}^{\infty} b_n x^n$ with all $b_n \ge 0$ and $\sum_n b_n = 1$.

Now, with the help of certain CCP functions and their inverses, the author is able to provide upper estimates for K_G; starting the procedure from the sign function produces Krivine's bound, and the complex sign function $\operatorname{sign}(z) = z/|z|$ (almost) leads to Haagerup's bound. (See Theorem 6.8 on page 161, Theorem 6.9 on page 166 and Example 6.7 on page 168; resp. Theorem 7.6 on page 189 and Example 7.1 on page 191.) It remains open how to choose other start functions that provide stronger estimates, although the approach shows that this might in theory be possible.

Apart from presenting the above strategy with all technical details (of which there are many), the author has interspersed his text with remarks, e.g. on Tsirelson's quantum correlation matrices, Bell's inequalities, the Grothendieck inequality in operator and tensor product theory, etc. A whole chapter is devoted to complex Gaussian random vectors that are at the heart of many arguments.

Chapter 8, outlining the approach at large, is very helpful. Still, I found the text a little difficult to navigate. For one thing, this results from the intrinsic difficulty and technical nature of the material; on the other hand, cross-referencing has not been made particularly easy for the readers since theorems, propositions, corollaries etc. have counters of their own (pretty much as in Grothendieck's opus magnum [*A. Grothendieck*, Produits tensoriels topologiques et espaces nucléaires. Providence, RI: American Mathematical Society (AMS) (1955; Zbl 0064.35501)]!). The index is a little scarce; for instance, the all-important entry correlation matrix is missing.

The late Professor Pietsch, in his "Operator ideals", has said, "Although nobody needs the exact value of Grothendieck's constant, everybody likes to know it." While there is no doubt about the second part of his sentiment, the first one can be considered dubious insofar as the hunt for sharper bounds involves new ideas and advances mathematics. The monograph at hand is testimony to this principle.

Reviewer: Dirk Werner (Berlin)

MSC:

46-02 Research exposition (monographs, survey articles) pertaining to functional analysis
15A60 Norms of matrices, numerical range, applications of functional analysis to matrix theory
46M05 Tensor products in functional analysis
47L20 Operator ideals
33C05 Classical hypergeometric functions, ${}_2F_1$
33C45 Orthogonal polynomials and functions of hypergeometric type (Jacobi, Laguerre, Hermite, Askey scheme, etc.)
41A58 Series expansions (e.g., Taylor, Lidstone series, but not Fourier series)
62H05 Characterization and structure theory for multivariate probability distributions; copulas
62H20 Measures of association (correlation, canonical correlation, etc.)
81P45 Quantum information, communication, networks (quantum-theoretic aspects)

Keywords:

Grothendieck inequality; Grothendieck constant; correlation matrices; Gaussian random vectors; hypergeometric functions; CCP functions

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